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A hyperbolic well-posed model for the flow of granular materials

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Abstract. A plasticity model for the flow of granular materials is presented which is derived from a physically based kinematic rule and which is closely related to the double-shearing model, the double-sliding free-rotating model and also to the plastic-potential model. All of these models incorporate various notions of the concept of rotation-rate and the crucial idea behind the model presented here is that it identifies this rotation-rate with a property associated with a Cosserat continuum, namely, the intrinsic spin. As a consequence of this identification, the stress tensor may become asymmetric. For simplicity, in the analysis presented here, the material parameters are assumed to be constant. The central results of the paper are that (a) the model is hyperbolic for two-dimensional steady-state flows in the inertial regime and (b) the model possesses a domain of linear well-posedness. Specifically, it is proved that incompressible flows are well-posed.

Key words: granular materials, hyperbolic, rigid-plastic, well-posed

1. Introduction

The last 10 to 15 years have seen a rapid increase in research effort into the study of granular materials. Such materials, and systems involving them, exhibit complex and interesting behaviour. Civil engineers have long had a need to model soils, in which the densely packed grains, with liquid and gas occupying the interstices, exhibit solid-like behaviour, while chemical engineers have needed to model fluid suspensions, liquid and gas, in which granular material is dispersed in the fluid. In recent years, these traditional areas involving practical engineering problems, have attracted the attention of physicists, who have developed an interest in such systems both to understand their fundamental behaviour and also for the purpose of analogy when investigating other complex systems. In addition to this, many industries (for example, the chemical and food industries) handle granular solids and both storage and bulk flow give rise to problems which have a direct adverse economic effect. One method of trying to solve these problems is by way of obtaining a better theoretical understanding of the underlying principles of the physics and mechanics of granular materials.

In this paper we shall consider a mass of granular material occupying a region of space and undergoing a flow or deformation. The grains are assumed to be densely packed, *i.e.*, each grain is in contact with several of its neighbours and contact with a neighbouring grain is of finite duration and non-impulsive (*i.e.*, contact between grains is not modelled as instantaneous impact). The kinematics of such systems have been remarkably resistant to successful mathematical modelling. There are a number of different types of models which may be summarised as (1) discrete, (2) statistical mechanical and (3) continuum, in which the first and third appear to be more appropriate for the very densely packed systems considered here. The statistical–mechanical theories are associated with the work of Jenkins, see for example [1]. Despite the popularity of discrete modelling based upon Newtonian mechanics in which the grains are modelled as small bodies and in which grain interactions may be modelled by including frictional, elastic and viscous effects in various ways, for large-scale systems computing power is still not adequate to solve those boundary- and initial-value problems of practical interest. For large systems the convenience of a continuum approach is a tempting goal, even though granular systems may appear to be at the borderline, or even beyond, that at which a continuum model may be considered applicable. Continuum models are also a useful framework within which to propose simple theories which may capture some aspect of the behaviour of the real material. And herein lies the problem: no continuum model has gained wide acceptance for its applicability to granular materials. Geotechnical and civil engineers usually use the so-called plastic-potential model, in which a yield condition (an algebraic inequality satisfied by the stress components) is assumed to hold in conjunction with the stress equilibrium equations and the strain increment or deformation-rate tensor is obtained by differentiation of the plastic potential with respect to the components of stress. If the yield condition and plastic potential are determined from the same function, the flow rule is called associated, otherwise it is called non-associated. Such plasticity models are now almost universal in soil mechanics; see for example the classic paper by Drucker and Prager [2], or, for a realistic plastic potential based upon careful experimentation, see [3], but they have not been adopted by researchers in any other field. It should be noted that there are dissenting voices even in geotechnical engineering, in particular there have been significant contributions due to G. Gudehus [4] and D. Kolymbas in the development of hypoplastic models; see [5]. An alternative class of models has been based directly upon physical arguments concerning the manner of flow of granular materials, but these again have not found widespread acceptance, finding support mainly from applied mathematicians. Historically, chemical engineers have studied systems in which the grains are in suspension, either in liquid or in a gas, and have thus treated the problem as one of fluid mechanics or rheology. Recently, physicists have conducted experiments on small-scale systems and are attempting to construct new types of model for such systems. It is also becoming more and more common to conduct computer simulations in lieu of performing real experiments and although there is something to be said in favour of such simulations, there is a danger that the distinction between simulation results and the results of real experiments is becoming blurred, particularly in view of the current fashion of referring to simulations as experiments! Discrete models, and the attendant simulations, have proponents in all disciplines, a key reference here is the classic paper by Cundall and Strack [6].

In this paper we wish to consider a unified plasticity theory based upon physical arguments. Such theories have had a number of theoretical problems associated with them, of which perhaps the most important is the loss of hyperbolicity and ill-posedness. Perhaps this latter term requires some explanation. The traditional definition of a well-posed problem is that it is one for which the solution exists, is unique and depends continuously on the boundary and initial data. An ill-posed problem, then, is merely the negation of this, one in which at least one of the above conditions fails to hold. Understood in this way, the phrase ill-posed problem can mean any one of a number of things, and in this sense is imprecise and not itself well (*i.e.*, uniquely) defined. However, we shall use the term in the following specific way. A set of partial differential equations is linearly ill-posed with respect to initial conditions if, given a solution to the initial-value problem, a sinusoidal perturbation of the given solution grows without bound in the limit of vanishingly short wavelengths. The basic facts concerning ill-posedness with regard to the models of granular materials were established in a number of important papers by Schaeffer and co-workers, [7–9], in which the method of frozen coefficients was used; see also, for a related paper, [10]. The implementation of the method of frozen coefficients used here is based upon that used in [11]. It may happen that the mathematical ill-posedness reflects some strong physical instability and in such a case the ill-posedness may be acceptable. But in the absence of a strong physical instability it seems more likely that equations admitting such a strong mathematical instability cannot be an acceptable mathematical model of the physical process. In this case we must look to some property in the physics or the mechanics of the problem that will regularise the equations. In the case of granular materials there appears to be no consensus as to whether ill-posed equations form an acceptable model. On the one hand, there is reluctance among many researchers to accept ill-posed theories, on the other hand the fact that granular materials do exhibit unstable behaviour gives credence to the possibility that they be modelled in some sense by an ill-posed theory. There also appear to be many researchers, particularly those working on quasi-static problems, who are oblivious to, or even ignore, the fact that the model they work with is ill-posed.

We shall construct a theory which incorporates the following physical properties and attributes. We regard these as the minimum necessary requirements to construct an acceptable theory from the point of view of both theoretical and physical properties. These are:

(a) a pressure-dependent yield condition which introduces the concept of internal friction and frictionally generated resistive stresses. A yield condition is an essential part of many plasticity models and seems well-founded physically, taking an analogy with the Coulomb law of dry friction from Newtonian particle and rigid-body mechanics;

(b) packed discrete grains that may deform, but certainly cannot overlap, can only flow by one grain overtaking another or by being overtaken, and this suggests that the basic continuum deformation is by shear;

(c) the impenetrability condition for the grains means that the grains have to re-arrange themselves, riding up and down over each other, to flow past one another. This rearrangement gives rise to dilatation, *i.e.*, to volume and bulk density changes;

(d) individual grains may rotate and the rotations combine to affect the macroscopic flow. This gives rise to a continuum rate of rotation phenomenon distinct from that of the anti-symmetric part of the velocity-gradient tensor (or vorticity).

Whether or not the above conditions are sufficient for a workable continuum model for granular materials is an open question but they appear all to be necessary for a complete theory capable of explaining sufficient physical facts. We regard the ill-posedness of the plastic-potential model as a sign that anything less is insufficient. The ill-posedness of the double-shearing model is due to the choice of the rate of rotation of the principal axes of stress as a measure of the rate of rotation described in (d). In this paper we replace this quantity by a physical angular velocity (intrinsic spin), a primitive kinematic quantity that has a rotational inertia and stress associated with it: the stress tensor, in general, may be non-symmetric.

As mentioned above, it is sometimes stated that ill-posedness of the governing equations is merely a reflection of the unstable nature of granular flow. But there has, as yet, been no convincing demonstration that the mathematical ill-posedness is associated with a particular physical instability. The nature of the ill-posedness is too strong, too all-pervading to be a reflection, say, of the growth of a single shear-band. In the authors' opinion, a more likely hypothesis is that the models are mathematically ill-posed due to a missing or incorrect physical law or attribute or due to the various physical laws being combined in an inconsistent way. We have stated above the physical properties that we wish to introduce into the model; all that remains is to express these laws in such a manner, and in such a combination, that the model is well-posed.

It is, perhaps, necessary to emphasise the following point. Plasticity models are mainly used in the quasi-static regime. But for an ill-posed plasticity model there is, strictly speaking, no such thing as a quasi-static regime. The inertia terms may be negligible in many practical problems, but they are not identically zero. For an ill-posed model, no matter how infinitesimally small the inertia terms are, they will grow increasingly large with time. For this reason, in this paper, where we demonstrate that our plasticity model is well-posed, it is essential that we retain the inertia terms in both the translational and rotational equations of motion.

Since no model within the confines of classical continuum mechanics has found general acceptance, the possibility must be faced that perhaps no such model exists. Indeed, many researchers have turned to the framework of a Cosserat continuum in order to regularise the equations, see for example [12]. However, such models are greatly complicated in comparison to standard models due to the presence of couple-stress. Further, the standard plastic-potential model and the double-shearing model both appear to capture successfully some aspects of the behaviour of granular materials. In this paper we wish to preserve as much as possible the relative simplicity of these latter models and hence wish to use a framework which extends the classical continuum as little as possible and to obtain equations as similar as possible to the plastic-potential and double-shearing models. As far as the authors are aware there is no conclusive experimental evidence for the existence of couple stress in granular materials and its introduction greatly complicates the equations. Moreover, the existence of grain rotation in granular materials is very much self-evident and yet researchers seem to have overlooked the possibility that incorporating some measure of grain rotation may itself be sufficient to both regularise the equations and to produce a model capable of explaining the major features of the bulk flow of granular materials. For this reason we introduce a Cosserat continuum in which there is no couple-stress but there is both rotation and rotational inertia and refer to this as a reduced Cosserat continuum.

For simplicity we shall consider a perfectly plastic model, *i.e.*, one in which the material parameters are constants and the conclusions presented here have been proved in this context. But it should be pointed out that the model may easily be extended to include density-hardening or softening, or indeed any other type of hardening or softening. In fact, since the material cannot undergo dilatation indefinitely, it is necessary for the parameter governing dilatation to reduce in magnitude to zero as the deformation or flow proceeds in order to obtain a realistic model. In this paper we shall ultimately avoid the question of the evolution of dilatancy and its effect on well-posedness by considering the important special case of incompressible flows.

2. The mathematical model

Consider a body $\mathcal{B}$ comprising a mass of granular material occupying a region $\mathcal{R}$ at time t in three-dimensional space and let $\mathcal{F}$ denote an inertial frame from which to observe $\mathcal{B}$. In the standard continuum model, the following field variables are defined at each spatial point P of $\mathcal{R}$. Relative to $\mathcal{F}$, let $\mathbf{v}$, σ, ρ denote the velocity, Cauchy stress and bulk density of the granular material at the material point of $\mathcal{B}$ instantaneously occupying the point P. We shall consider an enhanced continuum model, a type of Cosserat continuum, in which what is usually termed a material particle, in addition to the above, also possesses some of the attributes of a rigid body, namely an intrinsic spin ω and a moment of inertia $\mathbf{I}$, and for this reason is referred to as a material-point body. A standard Cosserat continuum possesses a further physical quantity, namely the couple-stress. However, couple-stress will not appear in our equations and so we refrain from introducing it into our continuum, and for this reason we call it a reduced Cosserat continuum. The quantity ω is an extra field variable and $\mathbf{I}$ represents a material property. As a result of their introduction the stress tensor will not, in general, be symmetric. In order to give an intuitive meaning to ω and $\mathbf{I}$, we may note that $\mathbf{I}$ is a second-order symmetric tensor and, as such, there exist three mutually perpendicular directions (which will not be unique if the principal values of inertia are not distinct) relative to which the matrix representation of $\mathbf{I}$ is diagonal. This triad of directions is fixed in the material-point body. During the deformation or flow the orientation of this triad will, in general, vary and ω is the spin or angular velocity of this triad.

In this paper we shall restrict consideration to planar flows of $\mathcal{B}$. Let $\mathbf{\Gamma}$ denote the velocity-gradient tensor with components defined by

$$\Gamma_{ij} = \frac{\partial v_i}{\partial x_j} \tag{1}$$

and define the deformation-rate tensor, $\mathbf{d}$, and the spin tensor, $\mathbf{s}$, with components

$$d_{ij} = \tfrac{1}{2}\left(\Gamma_{ij} + \Gamma_{ji}\right), \quad s_{ij} = \tfrac{1}{2}\left(\Gamma_{ij} - \Gamma_{ji}\right) \tag{2}$$

as its symmetric and anti-symmetric parts, respectively. Let Ox_i, Ox_i' denote two sets of rectangular Cartesian co-ordinate axes, in the plane of flow, with the latter inclined at an angle ϑ (measured anti-clockwise positive) to the former and denote the components of σ by σ_{ij}, σ_{ij}' relative to each set of axes, respectively, where the subscripts take the values 1 and 2. The stress components transform as

$$\begin{aligned}
\sigma_{11}' &= \tfrac{1}{2}\left[\sigma_{11} + \sigma_{22} + (\sigma_{11} - \sigma_{22})\cos 2\vartheta + (\sigma_{21} + \sigma_{12})\sin 2\vartheta\right],\\
\sigma_{12}' &= \tfrac{1}{2}\left[\sigma_{12} - \sigma_{21} - (\sigma_{11} - \sigma_{22})\sin 2\vartheta + (\sigma_{21} + \sigma_{12})\cos 2\vartheta\right],\\
\sigma_{21}' &= \tfrac{1}{2}\left[\sigma_{21} - \sigma_{12} - (\sigma_{11} - \sigma_{22})\sin 2\vartheta + (\sigma_{21} + \sigma_{12})\cos 2\vartheta\right],\\
\sigma_{22}' &= \tfrac{1}{2}\left[\sigma_{11} + \sigma_{22} - (\sigma_{11} - \sigma_{22})\cos 2\vartheta - (\sigma_{21} + \sigma_{12})\sin 2\vartheta\right].
\end{aligned} \tag{3}$$

It is convenient to define the invariant quantities

$$p_\sigma = -\tfrac{1}{2}(\sigma_{11} + \sigma_{22}), \quad r_\sigma = \tfrac{1}{2}(\sigma_{12} - \sigma_{21}), \quad q_\sigma = \tfrac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{21} + \sigma_{12})^2\right]^{1/2} \tag{4}$$

and to define ψ_σ, the angle that the greater principal direction of the *symmetric* part of the stress makes with the x_1-axis, by

$$\tan 2\psi_\sigma = \frac{\sigma_{21} + \sigma_{12}}{\sigma_{11} - \sigma_{22}} \tag{5}$$

and then

$$\begin{aligned}
&\sigma_{11} = -p_\sigma + q_\sigma \cos 2\psi_\sigma, \qquad \sigma_{22} = -p_\sigma - q_\sigma \cos 2\psi_\sigma,\\
&\sigma_{12} = r_\sigma + q_\sigma \sin 2\psi_\sigma, \qquad \sigma_{21} = -r_\sigma + q_\sigma \sin 2\psi_\sigma.
\end{aligned} \tag{6}$$

2.1. The stress equations of motion

Let ∂_t, ∂_i denote partial differentiation with respect to time t, x_i, $i = 1,2$, respectively; then the stress equations of motion are

$$\begin{aligned}
&\rho\partial_t v_1 + \rho v_1 \partial_1 v_1 + \rho v_2 \partial_2 v_1 - \partial_1 \sigma_{11} - \partial_2 \sigma_{21} - \rho F_1 = 0,\\
&\rho\partial_t v_2 + \rho v_1 \partial_1 v_2 + \rho v_2 \partial_2 v_2 - \partial_1 \sigma_{12} - \partial_2 \sigma_{22} - \rho F_2 = 0,
\end{aligned} \tag{7}$$

where F_1, F_2 denote the components of the body force per unit mass. From Equations (4) and (7) we obtain

$$\begin{aligned}&\rho\partial_t v_1+\rho v_1\partial_1 v_1+\rho v_2\partial_2 v_1+\partial_1 p_\sigma-\cos 2\psi_\sigma\partial_1 q_\sigma+2q_\sigma\sin 2\psi_\sigma\partial_1\psi_\sigma\\&\qquad+\partial_2 r_\sigma-\partial_2 q_\sigma\sin 2\psi_\sigma-2q_\sigma\cos 2\psi_\sigma\partial_2\psi_\sigma-\rho F_1=0,\\&\rho\partial_t v_2+\rho v_1\partial_1 v_2+\rho v_2\partial_2 v_2-\partial_1 r_\sigma-\sin 2\psi_\sigma\partial_1 q_\sigma-2q_\sigma\cos 2\psi_\sigma\partial_1\psi_\sigma\\&\qquad+\partial_2 p_\sigma+\partial_2 q_\sigma\cos 2\psi_\sigma-2q_\sigma\sin 2\psi_\sigma\partial_2\psi_\sigma-\rho F_2=0.\end{aligned} \tag{8}$$

The equation of rotational motion for a reduced Cosserat continuum may be written

$$\rho I\left(\partial_t\omega+v_1\partial_1\omega+v_2\partial_2\omega\right)+\rho\left(\partial_t I+v_1\partial_1 I+v_2\partial_2 I\right)\omega-2r_\sigma-\rho G=0, \tag{9}$$

where G is the body moment per unit mass. In this paper we assume that the moment-of-inertia tensor $\mathbf{I}$ is prescribed. At each point P, it represents, effectively, the moment of inertia of a representative volume element in the real material. A lower bound for this volume is the volume occupied by a single grain, but, in general, it will be a mesoscopic domain containing several grains and the attendant void space. In a standard Cosserat continuum, the equation of rotational motion would also contain the couple-stress.

The constitutive equation must be such as to allow the possibility of a non-symmetric state of stress. In this paper our intention is to propose a theory which is as similar as possible to standard plasticity models. Here, we shall adopt the simplest method of incorporating a possibly asymmetric stress tensor into the theory, namely, we shall assume that the symmetric and anti-symmetric parts of the stress may be treated separately, *i.e.*, they are independent of each other. We shall assume that the symmetric part of the stress satisfies a yield condition of the form

$$f\left(p_\sigma, q_\sigma, \rho\right)\leq 0, \tag{10}$$

where the function f is such that an angle of internal friction

$$\sin\phi=-f_p/f_q \tag{11}$$

may be defined, and where subscripts p, q indicate partial differentiation with respect to p_σ, q_σ, respectively. The simplest and most important yield condition of this type is the Mohr–Coulomb criterion, where if τ_n^s denotes the tangential traction associated with the symmetric part of the stress; then

$$\left|\tau_n^s\right|\leq-\sigma_n\tan\phi+k, \tag{12}$$

where σ_n denotes the normal component of traction across a surface with normal direction $\mathbf{n}$ and ϕ, k denote the angle of internal friction and cohesion, respectively. In invariant form the Coulomb yield condition may be written

$$q_\sigma\leq p_\sigma\sin\phi+k\cos\phi. \tag{13}$$

In addition, we introduce a rotational yield condition of the form

$$\left|r_\sigma\right|\leq m, \tag{14}$$

where $m>0$ is a material parameter, which we will call the rotational yield strength. We shall regard m as a new material parameter, independent of both ϕ and k. Note that (14) is not differentiable at $r_\sigma = 0$. It is possible for the stress tensor to be symmetric, $r_\sigma = 0$, and so we incorporate this case together with the case of equality in the inequality (14) by writing

$$r_\sigma = \epsilon m, \tag{15}$$

where ϵ may take one of the values $+1, 0, -1$ according as to whether r_σ is positive, zero or negative. Introducing a rotational yield condition in a rigid–plastic context brings with it the usual problem of indeterminacy, albeit in a slightly altered form. In classical plasticity the stress is indeterminate in the so-called rigid region, *i.e.*, in the region where $f(p_\sigma, q_\sigma, \rho) < 0$. In the case of a rotational yield condition, the asymmetric part of the stress is zero in the rotationally quasi-static regime. Whereas the linear momentum equations involve the divergence of the stress, allowing a multiplicity of solutions to the equilibrium equations, the balance of angular momentum involves the asymmetric stress directly, so a rotationally quasi-static flow leads to a symmetric stress tensor and to an indeterminacy, up to a constant, in the intrinsic spin field and hence also in the velocity field. We may call this the de Josselin de Jong indeterminacy, since it was first discovered by him in his double-sliding free-rotating model, [13]. The indeterminacy is removed in cases where there is a boundary condition to be satisfied by ω, or if physical arguments may be deployed to show that $\omega = 0$. An alternative way to remove the indeterminacy is to include rotational elasticity (presumably in the context of a full elastic-plastic model for the symmetric part of the stress as well) or by using rotational viscosity instead of the rotational yield condition. We will not pursue this idea in this paper and shall consider only the case of a rotational yield condition. Also, we wish to construct a generalisation of classical plasticity that is well-posed, but which is as similar to the classical case as possible, so we shall concentrate here on regions in which the translational yield condition is satisfied. Thus, the following regions are considered:

1. the indeterminate region $\mathcal{R}_i$, where one or both of $q_\sigma < p_\sigma \sin\phi + k\cos\phi$ and $r_\sigma \neq \epsilon m$ hold;
2. the fully deforming region $\mathcal{R}_d$, where $q_\sigma = p_\sigma \sin\phi + k\cos\phi$ and $r_\sigma = \epsilon m$.

Henceforth in this paper we shall consider only the region $\mathcal{R}_d$. In this region we may eliminate q_σ, r_σ from the stress equations of motion using the yield conditions (13) and (14) with equality holding, to obtain the equations

$$\begin{aligned}&\rho\partial_t v_1 + \rho v_1\partial_1 v_1 + \rho v_2\partial_2 v_1 + (1 - \sin\phi\cos 2\psi_\sigma)\,\partial_1 p_\sigma - \sin\phi\sin 2\psi_\sigma\partial_2 p_\sigma \\ &\quad + 2q_\sigma\sin 2\psi_\sigma\partial_1\psi_\sigma - 2q_\sigma\cos 2\psi_\sigma\partial_2\psi_\sigma - \rho F_1 = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\rho\partial_t v_2 + \rho v_1\partial_1 v_2 + \rho v_2\partial_2 v_2 - \sin\phi\sin 2\psi_\sigma\partial_1 p_\sigma + (1 + \sin\phi\cos 2\psi_\sigma)\,\partial_2 p_\sigma \\ &\quad - 2q_\sigma\cos 2\psi_\sigma\partial_1\psi_\sigma - 2q_\sigma\sin 2\psi_\sigma\partial_2\psi_\sigma - \rho F_2 = 0.\end{aligned} \tag{17}$$

Also, the continuity equation governing the evolution of the bulk density reads

$$\partial_t\rho + v_1\partial_1\rho + v_2\partial_2\rho + \rho\partial_1 v_1 + \rho\partial_2 v_2 = 0. \tag{18}$$

3. The kinematic equations

In this section we derive a pair of kinematic equations governing the flow of the ideal material, which are formally identical to the double-shearing equations due to Spencer [14] for incompressible materials and extended by Mehrabadi and Cowin [15] to dilatant materials. See also [16] for further developments of the model. The double-sliding free-rotating model, [17], was also extended to dilatant materials in a similar way to that of Mehrabadi and Cowin

[15]. However, one quantity occurring in the equations is here given a different interpretation to the above models. For a unified derivation of the double-shearing, double-sliding free-rotating and plastic-potential models, see [18]. It should be noted that single-shearing models have also been proposed; see for example [19–21], but it is difficult with such models to admit a sufficiently wide variety of flows. The derivation of the kinematic equations governing the flow for the model presented here is based on that of Harris [18].

Define the quantity ε by

$$\varepsilon = \pi/4 + \phi/2; \tag{19}$$

then at each point of $\mathcal{R}_d$, *i.e.*, the region in which the Coulomb yield condition is satisfied, it is well known that the inequality (12) is satisfied across line segments directed at angles of $\psi_\sigma + \varepsilon$, $\psi_\sigma - \varepsilon$ to the x_1-axis. We shall refer to these two directions, symmetric on either side of the major principal stress direction, as the α_1- and α_2-directions, respectively, or as the Coulomb yield directions associated with the inequality (12). Let $\mathbf{t}_{\alpha_i}$ denote a unit vector in the α_i-direction.

We shall also define a second pair of (non-coincident) directions which characterise the dilatancy of the material and which make angles of $\psi_\sigma - \varepsilon + \nu$, $\psi_\sigma + \varepsilon - \nu$ with the positive x_1-direction and will be referred to as the β_1-, β_2-directions, respectively. Thus, the β-directions are defined in terms of the α-directions and ν, where ν is called the *angle of dilatancy* and is to be regarded as a material parameter. In an incompressible material, $\nu = 0$ and in this case the β_1-(β_2-)direction coincides with the α_2-(α_1-)direction. Let $\mathbf{t}_{\beta_i}$ denote a unit vector in the β_i-direction.

Let P denote a material-point body in $\mathcal{R}_d$ and let Q denote a material-point body in an infinitesimal neighbourhood of P, with Q distinct from P. Let $\mathbf{ds}_P^Q$ denote the infinitesimal position vector of Q relative to P. Let Q_{α_i} denote the point of intersection of the α_i-direction through P with the α_j-direction through Q, where, here and in the sequel, $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. Consider the material body comprising the set of all material-point bodies in an infinitesimal neighbourhood of P and also, for fixed but arbitrary Q, consider the material body instantaneously occupying the infinitesimal parallelogram $PQ_{\alpha_1}Q_{\alpha_2}Q$. Let $\mathbf{dv}_P^{Q_{\alpha_i}}$ denote the velocity of the material point body at Q_{α_i} relative to that at P, $\mathbf{ds}_P^{Q_{\alpha_i}}$ denote the position vector of Q_{α_i} relative to P and let $\mathbf{n}_{\alpha_i}$ denote the normal to the α_i-direction (measured anti-clockwise positive from the direction $\mathbf{t}_{\alpha_i}$).

The kinematic model comprises two separate postulates. Firstly, the manner of the instantaneous rate of deformation of the material occupying the parallelogram $PQ_{\alpha_1}Q_{\alpha_2}Q$ in terms of the two adjacent sides PQ_{α_1}, PQ_{α_2} is specified by prescribing the velocities of the material-point bodies at Q_{α_1} and Q_{α_2} relative to P. Secondly, the manner in which the intrinsic spin gives rise to a velocity of Q relative to P is prescribed. It should be noted that this is a purely continuum hypothesis, no attempt being made to average the micro-mechanical velocities and rotations. The model is to be validated, or refuted, by comparison of its predictions with the properties of real granular materials.

3.1. The proposed kinematic hypothesis

The first postulate is that, relative to the inertial frame $\mathcal{F}$, the velocity of the material-point body at Q_{α_i} relative to P, $\mathbf{dv}_P^{Q_{\alpha_i}}$, is given by

$$\mathbf{dv}_P^{Q_{\alpha_i}} = k_{\beta_i} \left| \mathbf{ds}_P^{Q_{\alpha_i}} \right| \mathbf{t}_{\beta_i} \cos\phi, \tag{20}$$

where k_{β_i} is a proportionality factor called the *shear strength* in the β_i-direction and i takes the values 1 and 2. We shall call Equation (20) a β_i-direction dilatant shear on the α_i-direction, since both material-point bodies P and Q_{α_i} lie on the same α_i-line segment and their relative velocity is directed in the β_i-direction. For an arbitrary point Q in the neighbourhood of P the velocity of the material-point body at Q relative to that at P due to the two dilatant shears of Equation (20) is then postulated to be given by their sum

$$\mathbf{dv}_P^Q\Big|_{\mathcal{S}} = \mathbf{dv}_P^{Q_{\alpha_1}} + \mathbf{dv}_P^{Q_{\alpha_2}}, \tag{21}$$

relative to the inertial frame $\mathcal{F}$. The α_1- and α_2-directions are the shear directions and we shall also refer to them as the *slip directions.* A fundamental postulate of the model is that the slip directions coincide with the Coulomb yield directions. It should be carefully noted that Equations (20) and (21) hold at each instant of time. As the flow proceeds, the vectors $\mathbf{t}_{\alpha_i}$ and $\mathbf{t}_{\beta_i}$, $i=1,2$, will, in general, vary their orientation in space and, at each instant, the dilatant shears (20) are defined relative to the current orientations of the $\mathbf{t}_{\alpha_i}$, $\mathbf{t}_{\beta_i}$.

The second postulate is that, again relative to the inertial frame $\mathcal{F}$, the velocity of the material-point body at Q relative to that at P due to the intrinsic spin is the local rigid rotation

$$\mathbf{dv}_P^Q\Big|_{\mathcal{R}} = \omega \times \mathbf{ds}_P^Q. \tag{22}$$

Thus, the resultant relative velocity of Q relative to P, relative to the inertial frame $\mathcal{F}$, is

$$\mathbf{dv}_P^Q = \mathbf{dv}_P^Q\Big|_{\mathcal{S}} + \omega \times \mathbf{ds}_P^Q\big|_{\mathcal{R}}, \tag{23}$$

where ω is evaluated at P.

This is the complete kinematic rule for the local rate of deformation. Thus, for any point Q in the neighbourhood of P the velocity of the material point body at Q relative to that at P may be written

$$\mathbf{dv}_P^Q = k_{\beta_1}\left|\mathbf{ds}_P^{Q_{\alpha_1}}\right|\mathbf{t}_{\beta_1}\cos\phi + k_{\beta_2}\left|\mathbf{ds}_P^{Q_{\alpha_2}}\right|\mathbf{t}_{\beta_2}\cos\phi + \omega \times \mathbf{ds}_P^Q, \tag{24}$$

relative to $\mathcal{F}$.

3.2. The standard double-shearing kinematic hypothesis

It is instructive to compare the kinematic hypothesis given above, with that of the standard double-shearing model in terms of the method and notation of this paper. In order to state the hypothesis, a second frame of reference, this time non-inertial, is required. Let $\mathcal{G}$ denote a frame of reference fixed with respect to the principal axes of stress. Since the principal axes of stress may rotate, these axes form another spinning triad. Let ω_ψ denote the spin (angular velocity) of this triad relative to the inertial frame $\mathcal{F}$. The standard double-shearing model is also based upon Equations (20) and (21) but now we take these equations to be true relative to the frame $\mathcal{G}$. Since we require our constitutive equations to be expressed relative to the inertial frame $\mathcal{F}$, we must add the relative velocity $\mathbf{dv}_P^Q\Big|_{\mathcal{F}} = \omega_\psi \times \mathbf{ds}_P^Q$ due to the rigid spin ω_ψ of the frame $\mathcal{G}$ relative to $\mathcal{F}$. The complete kinematic hypothesis for the standard double-shearing model is then

$$\mathbf{dv}_P^Q = \mathbf{dv}_P^{Q_{\alpha_1}} + \mathbf{dv}_P^{Q_{\alpha_2}} + \omega_\psi \times \mathbf{ds}_P^Q \tag{25}$$

where the $\mathbf{dv}_P^{Q_{\alpha_i}}$ are again given by Equation (20). We see that the two kinematic hypotheses are formally identical, and will hence give rise to formally identical equations. In the planar case, where the orientation of the principal axes of stress in the plane of deformation is determined by ψ_σ, it is clear that ω_ψ is expressible in terms of the material time derivative of ψ_σ (material derivative since the material spins at this rate). Since the double-shearing equations have been derived from an equation essentially equivalent to Equation (25) in [18], we shall refrain from deriving them again here.

Now, the assertion that we observe two dilatant shears relative to axes fixed relative to the principal axes of the Cauchy stress tensor is equivalent to the assumption that, locally, the material rotates with these axes. This is an unusual, if not unique, assumption in continuum mechanics. It is, ultimately, this assumption that renders the double-shearing equations ill-posed, that causes the instability of a time-dependent simple shear, and is the cause of the discrepancy between the theory and certain experimental data; see [22] and [23]. The postulate that (20) and (21) are true relative to the frame $\mathcal{G}$ implies that the material rotates locally with the principal axes of stress and this is tantamount to postulating a physical law. It is this physical law that the authors identify as causing both the theoretical difficulties of ill-posedness and instability and also the discrepancy with the available experimental data. It is this physical law that the authors seek to replace with another law, the proposed kinematic hypothesis (25), which does not suffer from these disadvantages.

3.3. The proposed kinematic equations

We see then, that the essential difference between the standard double-shearing kinematic hypothesis and the one proposed here is that one spin of a triad, the spin of the principal axes of stress, is replaced by the spin of another triad, the spin of the principal axes of inertia. The proposed kinematic hypothesis is more in keeping with the double-sliding free-rotating model due to de Josselin de Jong [2,17]. However, this latter model is indeterminate, consisting, in one formulation, of a system of equations with more unknowns than equations (the unknown not matched with an equation being ω) and in a second formulation, of a set of inequalities. The standard double-shearing model represents one method of closing the double-sliding free-rotating equations, by prescribing the rotation to coincide with the rotation of the principal axes of stress. We may view the model presented here as an alternative closure of the double-sliding free-rotating equations, in which the rotation is identified with the spin of the material-point body and the system is closed by the equation of rotational motion. We turn now to the derivation of the equations arising from the proposed kinematic hypothesis (24). Now the angle between the α_1-direction and the normal to the α_2-direction is

$$\psi_\sigma + \varepsilon - (\psi_\sigma - \varepsilon + \pi/2) = \phi, \tag{26}$$

while the angle between the α_2-direction and the normal to the α_1-direction is

$$\psi_\sigma - \varepsilon - (\psi_\sigma + \varepsilon + \pi/2) = -\phi - \pi. \tag{27}$$

Hence the projections of $\mathbf{ds}_P^{Q\alpha_i}$ and $\mathbf{ds}_P^Q$ onto the direction $\mathbf{n}_{\alpha_j}$ are equal and may be written as

$$\left|\mathbf{ds}_P^{Q\alpha_1}\right| \cos\phi = \mathbf{n}_{\alpha_2} \cdot \mathbf{ds}_P^Q, \quad \left|\mathbf{ds}_P^{Q\alpha_2}\right| \cos\phi = -\mathbf{n}_{\alpha_1} \cdot \mathbf{ds}_P^Q. \tag{28}$$

Hence

$$\mathbf{dv}_P^Q = k_{\beta_1} \left(\mathbf{n}_{\alpha_2} \cdot \mathbf{ds}_P^Q\right) \mathbf{t}_{\beta_1} - k_{\beta_2} \left(\mathbf{n}_{\alpha_1} \cdot \mathbf{ds}_P^Q\right) \mathbf{t}_{\beta_2} + \omega \times \mathbf{ds}_P^Q, \tag{29}$$

i.e.,

$$\mathbf{dv}_P^Q = \left[k_{\beta_1} \left(\mathbf{t}_{\beta_1} \otimes \mathbf{n}_{\alpha_2}\right) - k_{\beta_2} \left(\mathbf{t}_{\beta_2} \otimes \mathbf{n}_{\alpha_1}\right) + \Omega\right] \cdot \mathbf{ds}_P^Q, \tag{30}$$

where Ω denotes the anti-symmetric tensor dual to ω. We now suppose that the velocity and rotational fields are sufficiently smooth in terms of the velocity-gradient tensor

$$\mathbf{dv}_P^Q = \boldsymbol{\Gamma} \cdot \boldsymbol{ds}_P^Q \tag{31}$$

and so

$$\boldsymbol{\Gamma} = k_{\beta_1}\left(\mathbf{t}_{\beta_1} \otimes \mathbf{n}_{\alpha_2}\right) - k_{\beta_2}\left(\mathbf{t}_{\beta_2} \otimes \mathbf{n}_{\alpha_1}\right) + \Omega, \tag{32}$$

where

$$\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}. \tag{33}$$

Now,

$$\begin{aligned} &\mathbf{t}_{\beta_1} = (\cos(\psi_\sigma - \varepsilon + \nu), \sin(\psi_\sigma - \varepsilon + \nu)), \quad \mathbf{t}_{\beta_2} = (\cos(\psi_\sigma + \varepsilon - \nu), \sin(\psi_\sigma + \varepsilon - \nu)), \\ &\mathbf{n}_{\alpha_1} = (-\sin(\psi_\sigma + \varepsilon), \cos(\psi_\sigma + \varepsilon)), \quad \mathbf{n}_{\alpha_2} = (-\sin(\psi_\sigma - \varepsilon), \cos(\psi_\sigma - \varepsilon)) \end{aligned} \tag{34}$$

and

$$\mathbf{t}_{\beta_1} \otimes \mathbf{n}_{\alpha_2} = \begin{bmatrix} -\cos(\psi_\sigma - \varepsilon + \nu)\sin(\psi_\sigma - \varepsilon) & \cos(\psi_\sigma - \varepsilon + \nu)\cos(\psi_\sigma - \varepsilon) \\ -\sin(\psi_\sigma - \varepsilon + \nu)\sin(\psi_\sigma - \varepsilon) & \sin(\psi_\sigma - \varepsilon + \nu)\cos(\psi_\sigma - \varepsilon) \end{bmatrix}, \tag{35}$$

$$\mathbf{t}_{\beta_2} \otimes \mathbf{n}_{\alpha_1} = \begin{bmatrix} -\cos(\psi_\sigma + \varepsilon - \nu)\sin(\psi_\sigma + \varepsilon) & \cos(\psi_\sigma + \varepsilon - \nu)\cos(\psi_\sigma + \varepsilon) \\ -\sin(\psi_\sigma + \varepsilon - \nu)\sin(\psi_\sigma + \varepsilon) & \sin(\psi_\sigma + \varepsilon - \nu)\cos(\psi_\sigma + \varepsilon) \end{bmatrix}. \tag{36}$$

Hence

$$\begin{aligned} \partial_1 v_1 &= -k_{\beta_1}\cos(\psi_\sigma - \varepsilon + \nu)\sin(\psi_\sigma - \varepsilon) + k_{\beta_2}\cos(\psi_\sigma + \varepsilon - \nu)\sin(\psi_\sigma + \varepsilon), \\ \partial_2 v_1 &= k_{\beta_1}\cos(\psi_\sigma - \varepsilon + \nu)\cos(\psi_\sigma - \varepsilon) - k_{\beta_2}\cos(\psi_\sigma + \varepsilon - \nu)\cos(\psi_\sigma + \varepsilon) - \omega, \\ \partial_1 v_2 &= -k_{\beta_1}\sin(\psi_\sigma - \varepsilon + \nu)\sin(\psi_\sigma - \varepsilon) + k_{\beta_2}\sin(\psi_\sigma + \varepsilon - \nu)\sin(\psi_\sigma + \varepsilon) + \omega, \\ \partial_2 v_2 &= k_{\beta_1}\sin(\psi_\sigma - \varepsilon + \nu)\cos(\psi_\sigma - \varepsilon) - k_{\beta_2}\sin(\psi_\sigma + \varepsilon - \nu)\cos(\psi_\sigma + \varepsilon), \end{aligned}$$

from which we obtain

$$\begin{aligned} \partial_1 v_1 + \partial_2 v_2 &= \left(k_{\beta_1} + k_{\beta_2}\right)\sin\nu, \\ \partial_1 v_1 - \partial_2 v_2 &= -k_{\beta_1}\sin(2\psi_\sigma - 2\varepsilon + \nu) + k_{\beta_2}\sin(2\psi_\sigma + 2\varepsilon - \nu) \\ &= k_{\beta_1}\cos(2\psi_\sigma - \phi + \nu) + k_{\beta_2}\cos(2\psi_\sigma + \phi - \nu), \\ \partial_2 v_1 + \partial_1 v_2 &= k_{\beta_1}\cos(2\psi_\sigma - 2\varepsilon + \nu) - k_{\beta_2}\cos(2\psi_\sigma + 2\varepsilon - \nu) \\ &= k_{\beta_1}\sin(2\psi_\sigma - \phi + \nu) + k_{\beta_2}\sin(2\psi_\sigma + \phi - \nu), \\ \partial_2 v_1 - \partial_1 v_2 &= \left(k_{\beta_1} - k_{\beta_2}\right)\cos\nu - 2\omega. \end{aligned} \tag{37}$$

From Equations $(37)_{\text{ii}}$ and $(37)_{\text{iii}}$ we may solve for the quantities k_{β_1} and k_{β_2} in terms of the components of the deformation-rate tensor to obtain

$$\begin{aligned} (d_{11} - d_{22})\sin(2\psi_\sigma + \phi - \nu) - 2d_{12}\cos(2\psi_\sigma + \phi - \nu) &= k_{\beta_1}\sin 2(\phi - \nu), \\ (d_{11} - d_{22})\sin(2\psi_\sigma - \phi + \nu) - 2d_{12}\cos(2\psi_\sigma - \phi + \nu) &= -k_{\beta_2}\sin 2(\phi - \nu), \end{aligned} \tag{38}$$

provided $\nu \neq \phi$. Subtracting and adding Equations (38) gives

$$\begin{aligned} \left(k_{\beta_1} + k_{\beta_2}\right)\sin 2(\phi - \nu) &= (d_{11} - d_{22})[\sin(2\psi_\sigma + \phi - \nu) - \sin(2\psi_\sigma - \phi + \nu)] \\ &\quad + 2d_{12}[\cos(2\psi_\sigma - \phi + \nu) - \cos(2\psi_\sigma + \phi - \nu)] \\ &= 2[(d_{11} - d_{22})\cos 2\psi_\sigma + 2d_{12}\sin 2\psi_\sigma]\sin(\phi - \nu), \end{aligned}$$

i.e.,

$$\left(k_{\beta_1}+k_{\beta_2}\right)\cos(\phi-\nu)=(d_{11}-d_{22})\cos 2\psi_\sigma+2d_{12}\sin 2\psi_\sigma \tag{39}$$

and

$$\begin{aligned}\left(k_{\beta_1}-k_{\beta_2}\right)\sin 2(\phi-\nu)&=(d_{11}-d_{22})[\sin(2\psi_\sigma+\phi-\nu)+\sin(2\psi_\sigma-\phi+\nu)]\\&\quad-2d_{12}[\cos(2\psi_\sigma+\phi-\nu)+\cos(2\psi_\sigma-\phi+\nu)]\\&=2[(d_{11}-d_{22})\sin 2\psi_\sigma-2d_{12}\cos 2\psi_\sigma]\cos(\phi-\nu),\end{aligned}$$

i.e.,

$$\left(k_{\beta_1}-k_{\beta_2}\right)\sin(\phi-\nu)=(d_{11}-d_{22})\sin 2\psi_\sigma-2d_{12}\cos 2\psi_\sigma. \tag{40}$$

Eliminating the quantities $k_{\beta_1}+k_{\beta_2}$ and $k_{\beta_1}-k_{\beta_2}$, between Equations $(37)_{\text{i}}$, $(37)_{\text{iv}}$, (39) and (40) gives the following pair of equations

$$d_{11}+d_{22}=\frac{\sin\nu}{\cos(\phi-\nu)}[(d_{11}-d_{22})\cos 2\psi_\sigma+2d_{12}\sin 2\psi_\sigma], \tag{41}$$

$$2(\omega-s_{21})=\frac{\cos\nu}{\sin(\phi-\nu)}[(d_{11}-d_{22})\sin 2\psi_\sigma-2d_{12}\cos 2\psi_\sigma]. \tag{42}$$

Equations (41) and (42) are the required constitutive equations governing the flow. They are formally identical to the Mehrabadi–Cowin equations [15]; however, the quantity ω is here interpreted as the intrinsic spin, whereas in the Mehrabadi–Cowin equations it is interpreted as the material derivative of the quantity ψ_σ.

Equations (41) and (42) are frame-indifferent. To see this, we demonstrate that they are unchanged in form under a superposed rigid-body motion. Consider two velocity and intrinsic spin fields (v_1, v_2, ω), $(v_1^{(1)}, v_2^{(1)}, \omega^{(1)})$, differing only by a rigid-body spin $(0, 0, \Omega)$, measured anti-clockwise positive, then substituting

$$v_1=v_1^{(1)}-\Omega x_2,\quad v_2=v_2^{(1)}+\Omega x_1,\quad \omega=\omega^{(1)}+\Omega$$

in Equations (41) and (42) gives the required result.

In the case where $\phi=\nu$, the first three of Equations (37) are equivalent to those for an associated flow rule, see [2],

$$d_{11}+d_{22}=\sin\phi[(d_{11}-d_{22})\cos 2\psi_\sigma+2d_{12}\sin 2\psi_\sigma], \tag{43}$$

$$(d_{11}-d_{22})\sin 2\psi_\sigma-2d_{12}\cos 2\psi_\sigma=0. \tag{44}$$

Equation (44) is the statement of coaxiality of the stress and deformation-rate tensors. Equations (41) and (42) also reduce to Equations (43) and (44) when $\phi=\nu$. In this case the velocity field is independent of ω and the quantity $k_{\beta_1}-k_{\beta_2}$ is indeterminate. Equation $(37)_{\text{iv}}$ then also becomes indeterminate unless we make the additional assumption that $k_{\beta_1}=k_{\beta_2}$ and then $(37)_{\text{iv}}$ reduces to

$$\omega=s_{21}, \tag{45}$$

i.e., the intrinsic spin ω is determined by the velocity field and is equal to half the vorticity. The anti-symmetric part of the stress required to ensure satisfaction of this kinematic constraint is obtained from the equation of rotational motion; Equation (9) and, in this case, the yield condition (14) must be omitted from the model, since the material must be able to sustain the anti-symmetric stress required in order to ensure satisfaction of Equation (45). In this sense, the model links, inextricably, the concepts of dilatancy, coaxiality, intrinsic spin and vorticity. Since it is an experimental fact that the magnitude of the angle of dilatancy is less than the angle of internal friction, it follows necessarily that the model predicts both non-coaxiality and non-coincidence of the intrinsic spin with half the vorticity, the rotational yield condition limiting the magnitude of the anti-symmetric part of the stress.

In summary, the model proposed in this section, expressed mathematically in Equations (41) and (42), may be described in physical terms by saying that the flow consists of a local intrinsic spin together with simultaneous dilatant shears on two slip directions and, further, these slip directions coincide with the Coulomb yield directions. The derivation is essentially algebraic, depending only on the yield condition and is independent of the equations of motion. When the full equations of the model are considered as a set of first-order partial differential equations, the question naturally arises as to the relationship, if any, between the slip and yield directions on the one hand and the spatial characteristic directions for a steady-state motion, on the other. We show in the next section that for this model there are three distinct spatial characteristic directions, two of which coincide with the coincident slip and yield directions, while the remaining characteristic direction corresponds to the direction of the streamlines.

It turns out that the well- or ill-posedness of this system of first-order partial differential equations is dependent upon certain properties of the characteristic directions. In this way, the mechanical and kinematic concepts of yield and slip directions are directly related to the mathematical concept of characteristic direction, and hence to the well- or ill-posedness of the model.

4. The steady-state equations are hyperbolic

For steady-state flows in the Ox_1x_2-plane, the equations governing the model (9), (16–18), (41), (42) become,

$$\begin{aligned}&\rho v_1\partial_1 v_1+\rho v_2\partial_2 v_1+(1-\sin\phi\cos 2\psi_\sigma)\,\partial_1 p_\sigma-\sin\phi\sin 2\psi_\sigma\partial_2 p_\sigma\\&\qquad+2q_\sigma\sin 2\psi_\sigma\partial_1\psi_\sigma-2q_\sigma\cos 2\psi_\sigma\partial_2\psi_\sigma-\rho F_1=0,\end{aligned}\tag{46}$$

$$\begin{aligned}&\rho v_1\partial_1 v_2+\rho v_2\partial_2 v_2-\sin\phi\sin 2\psi_\sigma\partial_1 p_\sigma+(1+\sin\phi\cos 2\psi_\sigma)\,\partial_2 p_\sigma\\&\quad-2q_\sigma\cos 2\psi_\sigma\partial_1\psi_\sigma-2q_\sigma\sin 2\psi_\sigma\partial_2\psi_\sigma-\rho F_2=0,\end{aligned}\tag{47}$$

$$\rho I\,(v_1\partial_1\omega+v_2\partial_2\omega)+\rho\,(v_1\partial_1 I+v_2\partial_2 I)\,\omega-2r_\sigma-\rho G=0,\tag{48}$$

$$v_1\partial_1\rho+v_2\partial_2\rho+\rho\partial_1 v_1+\rho\partial_2 v_2=0,\tag{49}$$

$$\begin{aligned}&[\cos(\phi-\nu)-\sin\nu\cos 2\psi_\sigma]\,\partial_1 v_1-(\sin\nu\sin 2\psi_\sigma)\,\partial_2 v_1-(\sin\nu\sin 2\psi_\sigma)\,\partial_1 v_2\\&\quad+[\cos(\phi-\nu)+\sin\nu\cos 2\psi_\sigma]\,\partial_2 v_2=0,\end{aligned}\tag{50}$$

$$\begin{aligned}&-(\cos\nu\sin 2\psi_\sigma)\,\partial_1 v_1+[\sin(\phi-\nu)+\cos\nu\cos 2\psi_\sigma]\,\partial_2 v_1\\&\quad+[-\sin(\phi-\nu)+\cos\nu\cos 2\psi_\sigma]\,\partial_1 v_2+(\cos\nu\sin 2\psi_\sigma)\,\partial_2 v_2+2\omega\sin(\phi-\nu)=0.\end{aligned}\tag{51}$$

Let

$$\mathbf{z}^t=(z_1,z_2,z_3,z_4,z_5,z_6)=(v_1,v_2,\omega,\rho,p_\sigma,\psi_\sigma)\,,\tag{52}$$

where the superscript t denotes transpose; then Equations (46–51) may be written in matrix form,

$$B_1(\mathbf{z})\,\partial_1\mathbf{z} + B_2(\mathbf{z})\,\partial_2\mathbf{z} + \mathbf{c}(\mathbf{z}) = 0, \tag{53}$$

where

$$B_1(\mathbf{z}) = \begin{bmatrix} \rho v_1 & 0 & 0 & 0 & b^1_{15} & b^1_{16} \\ 0 & \rho v_1 & 0 & 0 & b^1_{25} & b^1_{26} \\ 0 & 0 & \rho I v_1 & 0 & 0 & 0 \\ \rho & 0 & 0 & v_1 & 0 & 0 \\ b^1_{51} & b^1_{52} & 0 & 0 & 0 & 0 \\ b^1_{61} & b^1_{62} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{54}$$

$$B_2(\mathbf{z}) = \begin{bmatrix} \rho v_2 & 0 & 0 & 0 & b^2_{15} & b^2_{16} \\ 0 & \rho v_2 & 0 & 0 & b^2_{25} & b^2_{26} \\ 0 & 0 & \rho I v_2 & 0 & 0 & 0 \\ 0 & \rho & 0 & v_2 & 0 & 0 \\ b^2_{51} & b^2_{52} & 0 & 0 & 0 & 0 \\ b^2_{61} & b^2_{62} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{55}$$

$$\mathbf{c}(\mathbf{z}) = \begin{bmatrix} -\rho F_1 \\ -\rho F_2 \\ \rho \dot{I}\omega - 2r_\sigma - \rho G \\ 0 \\ 0 \\ 2\omega \sin(\phi - \nu) \end{bmatrix}, \tag{56}$$

where the superposed dot denotes the material derivative and where

$$\begin{aligned} b^1_{15} &= 1 - \sin\phi \cos 2\psi_\sigma, & b^1_{16} &= 2q_\sigma \sin 2\psi_\sigma, \\ b^1_{25} &= -\sin\phi \sin 2\psi_\sigma, & b^1_{26} &= -2q_\sigma \cos 2\psi_\sigma, \\ b^1_{51} &= \cos(\phi - \nu) - \sin\nu \cos 2\psi_\sigma, & b^1_{52} &= -\sin\nu \sin 2\psi_\sigma, \\ b^1_{61} &= -\cos\nu \sin 2\psi_\sigma, & b^1_{62} &= -\sin(\phi - \nu) + \cos\nu \cos 2\psi_\sigma, \end{aligned} \tag{57}$$

$$\begin{aligned} b^2_{15} &= -\sin\phi \sin 2\psi_\sigma, & b^2_{16} &= -2q_\sigma \cos 2\psi_\sigma, \\ b^2_{25} &= 1 + \sin\phi \cos 2\psi_\sigma, & b^2_{26} &= -2q_\sigma \sin 2\psi_\sigma, \\ b^2_{51} &= -\sin\nu \sin 2\psi_\sigma, & b^2_{52} &= \cos(\phi - \nu) + \sin\nu \cos 2\psi_\sigma, \\ b^2_{61} &= \sin(\phi - \nu) + \cos\nu \cos 2\psi_\sigma & b^2_{62} &= \cos\nu \sin 2\psi_\sigma. \end{aligned} \tag{58}$$

Let $u = u(x_1, x_2) = c$, where c is a constant, denote a curve in the Ox_1x_2-plane on which the solution $\mathbf{z}$ is known and define

$$\xi_1 = \partial_1 u, \quad \xi_2 = \partial_2 u. \tag{59}$$

Then $\mathbf{z} = \mathbf{z}(u) = \mathbf{z}(x_1, x_2)$ and, letting $d_u\mathbf{z}$ denote differentiation with respect to u, we have

$$\partial_1\mathbf{z} = d_u\mathbf{z}\,\partial_1 u = \xi_1 d_u\mathbf{z}, \quad \partial_2\mathbf{z} = d_u\mathbf{z}\,\partial_2 u = \xi_2 d_u\mathbf{z}. \tag{60}$$

Since $d_u\mathbf{z}=0$ on the curve $u(x_1, x_2)=c$, we may regard $d_u\mathbf{z}$ as an exterior derivative. Let

$$E = B_1\xi_1 + B_2\xi_2; \tag{61}$$

then

$$E = \begin{bmatrix} \rho e & 0 & 0 & 0 & e_{15} & e_{16} \\ 0 & \rho e & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & \rho I e & 0 & 0 & 0 \\ \rho\xi_1 & \rho\xi_2 & 0 & e & 0 & 0 \\ e_{51} & e_{52} & 0 & 0 & 0 & 0 \\ e_{61} & e_{62} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{62}$$

where

$$e = v_1\xi_1 + v_2\xi_2, \quad e_{ij} = b^1_{ij}\xi_1 + b^2_{ij}\xi_2. \tag{63}$$

Using Equations (60) and (61), we may write Equation (53) as

$$E d_u\mathbf{z} + \mathbf{c} = \mathbf{0}. \tag{64}$$

This is a set of equations for the exterior derivative $d_u\mathbf{z}$ which enables the solution $\mathbf{z}$ to be continued into the Ox_1x_2-plane away from the curve $u(x_1, x_2)=c$. Equation (64) fails to have a solution, *i.e.*, $\mathbf{z}$ cannot be continued off the curve $u(x_1, x_2)=c$, if

$$\det E = 0. \tag{65}$$

This is the condition that $u(x_1, x_2)=c$ be a characteristic curve. Expanding the determinant gives

$$\det E = \rho I e^2 (e_{26}e_{15} - e_{16}e_{25})(e_{51}e_{62} - e_{61}e_{52}), \tag{66}$$

where

$$e_{26}e_{15} - e_{16}e_{25} = -2q_\sigma A(\xi_1, \xi_2), \tag{67}$$

$$e_{51}e_{62} - e_{61}e_{52} = \cos(\phi - 2\nu) A(\xi_1, \xi_2) \tag{68}$$

and

$$A = A(\xi_1, \xi_2) = (\cos 2\psi_\sigma - \sin\phi)\xi_1^2 + 2\sin 2\psi_\sigma \xi_1\xi_2 - (\cos 2\psi_\sigma + \sin\phi)\xi_2^2. \tag{69}$$

Thus,

$$\det E = -2q_\sigma \rho I \cos(\phi - 2\nu) e^2 A^2 \tag{70}$$

and we note that the contributions to $\det E$ from the stress and kinematic equations uncouple and so we may refer to the two pairs of characteristic curves arising from Equations (67) and (68) as the stress and velocity characteristic curves, respectively. Further, their contributions are identical up to a multiplicative factor, and hence these curves coincide. All the characteristic curves are given by the condition (65) and hence one of the following must hold

$$v_1\xi_1 + v_2\xi_2 = 0, \tag{71}$$

$$(\cos 2\psi_\sigma - \sin\phi)\xi_1^2 + (2\sin 2\psi_\sigma)\xi_1\xi_2 - (\cos 2\psi_\sigma + \sin\phi)\xi_2^2 = 0. \tag{72}$$

Thus, the system has
1. a repeated characteristic linear in ξ_1, ξ_2,
2. a repeated pair of characteristic curves, quadratic in ξ_1 and ξ_2,

and, the system is hyperbolic in the sense that all characteristic directions are real, albeit degenerate in that each characteristic is repeated. Let $u = u(x_1, x_2) = c$ be a characteristic curve, then the condition

$$du = \xi_1 dx_1 + \xi_2 dx_2 = 0 \tag{73}$$

gives

$$\frac{dx_2}{dx_1} = -\frac{\xi_1}{\xi_2} = m_i, \tag{74}$$

say, where i takes the values 1,2,3. Hence, the root of the linear equation may be written $m_3 = v_2/v_1$, and the characteristic direction determined by this equation corresponds to the streamlines of the flow, while the roots of the quadratic equation

$$(\cos 2\psi_\sigma - \sin\phi)\, m_i^2 - (2\sin 2\psi_\sigma)\, m_i - (\cos 2\psi_\sigma + \sin\phi) = 0 \tag{75}$$

determine the characteristic directions in the Ox_1x_2 plane given by

$$m_1 = \tan(\psi_\sigma + \varepsilon), \quad m_2 = \tan(\psi_\sigma - \varepsilon). \tag{76}$$

These are the α_1- and α_2-directions defined after Equation (19) and the angle between them is $\frac{1}{2}\pi + \phi$. Thus, the characteristic directions of the system of governing partial differential equations coincide with the Coulomb yield directions and the slip directions. In fact, for equations of the form considered here, the condition that the stress and velocity characteristic directions coincide is a necessary, but not sufficient condition, for the linear well-posedness of the model. The plastic-potential model for incompressible materials does not have coincident stress and velocity characteristics and is ill-posed. On the other hand, the incompressible double-shearing model does have coincident stress and velocity characteristics but is also ill-posed. In the next two sections we demonstrate that the model proposed here, which is closely related to both the plastic potential and to the double-shearing models does admit a class of flows for which the model is linearly well-posed, namely the class of flows for an incompressible material.

5. Linearisation of the equations in the incompressible case

We now begin the proof that the model contains a domain of well-posedness. The calculation for the full model is extremely lengthy and requires specification of the evolution of the angle of dilatancy ν, since the standard assumption of perfect plasticity in which ν is considered constant is not adequate here. The total amount of dilatancy or compressibility must be limited, for otherwise the model becomes invalid, either as the density reduces below the level at which the grains can remain in contact, or as it increases above the level which requires grain overlap. Unlimited dilatancy, in particular, may give rise to a mathematical ill-posedness, in this case valid and caused by a physical instability, namely the phase change from solid or liquid-like behaviour to gaseous-like behaviour. The model proposed here, of course, becomes invalid for such dilute flows. Consequently, we consider the special case of incompressible flows and put $\nu = 0$ and omit ρ from the set of dependent variables. In this case, Equation (41) becomes identical with the continuity equation for an incompressible material and so we omit the latter. The full time-dependent equations for incompressible flows will be written as

$$A\partial_t \mathbf{z} + B_1(\mathbf{z})\,\partial_1 \mathbf{z} + B_2(\mathbf{z})\,\partial_2 \mathbf{z} + \mathbf{c}(\mathbf{z}) = 0, \tag{77}$$

where

$$A = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{78}$$

and B_1, B_2 are as in the previous section but with the fourth row and column omitted. We now linearise this set of equations. Let $\mathbf{Z^t} = (Z_1, Z_2, Z_3, Z_4, Z_5) = (V_1, V_2, \Omega, P_\sigma, \Psi_\sigma)$ denote a known solution of Equation (77) and consider a perturbation $\mathbf{z}'$ of $\mathbf{Z}$ such that

$$\mathbf{z} = \mathbf{Z} + \mathbf{z}'. \tag{79}$$

Also let Q_σ, R_σ denote the known q_σ, r_σ fields for this solution, q'_σ, r'_σ the corresponding perturbations of the q_σ, r_σ fields, then

$$\begin{aligned} &v_1 = V_1 + v'_1, \quad v_2 = V_2 + v'_2, \quad \omega = \Omega + \omega', \quad p_\sigma = P_\sigma + p'_\sigma, \\ &\psi_\sigma = \Psi_\sigma + \psi'_\sigma, \quad q_\sigma = Q_\sigma + q'_\sigma, \quad r_\sigma = R_\sigma + r'_\sigma, \end{aligned} \tag{80}$$

Finally, let D_{ij}, Σ_{ij} denote the components of the deformation-rate and stress tensors in the underlying unperturbed solution. Substituting Equation (79) in Equation (77), linearizing the resulting equations and using the fact that $\mathbf{Z}$ is a solution of (77), we have

$$A\partial_t \mathbf{z}' + B_1(\mathbf{Z})\,\partial_1 \mathbf{z}' + B_2(\mathbf{Z})\,\partial_2 \mathbf{z}' + \mathbf{C}(\mathbf{Z})\,\mathbf{z}' = 0, \tag{81}$$

where

$$\mathbf{C}(\mathbf{Z}) = \begin{bmatrix} c_{11} & c_{12} & 0 & c_{14} & c_{15} \\ c_{21} & c_{22} & 0 & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{53} & 0 & c_{55} \end{bmatrix} \tag{82}$$

and

$$\begin{aligned} &c_{11} = \rho_0 \partial_1 V_1, \quad c_{12} = \rho_0 \partial_2 V_1, \\ &c_{14} = 2\sin\phi\,(\partial_1 \Psi_\sigma \sin 2\Psi_\sigma - \partial_2 \Psi_\sigma \cos 2\Psi_\sigma), \\ &c_{15} = 2\,[\sin\phi\,(\partial_1 P_\sigma \sin 2\Psi_\sigma - \partial_2 P_\sigma \cos 2\Psi_\sigma) \\ &\qquad + 2Q_\sigma\,(\partial_1 \Psi_\sigma \cos 2\Psi_\sigma + \partial_2 \Psi_\sigma \sin 2\Psi_\sigma)], \\ &c_{21} = \rho_0 \partial_1 V_2, \quad c_{22} = \rho_0 \partial_2 V_2, \\ &c_{24} = -2\sin\phi\,(\partial_1 \Psi_\sigma \cos 2\Psi_\sigma + \partial_2 \Psi_\sigma \sin 2\Psi_\sigma), \\ &c_{25} = -2\,[\sin\phi\,(\partial_1 P_\sigma \cos 2\Psi_\sigma + \partial_2 P_\sigma \sin 2\Psi_\sigma) \\ &\qquad - 2Q_\sigma\,(\partial_1 \Psi_\sigma \sin 2\Psi_\sigma - \partial_2 \Psi_\sigma \cos 2\Psi_\sigma)], \\ &c_{31} = \rho I \partial_1 \Omega, \quad c_{32} = \rho I \partial_2 \Omega, \quad c_{33} = \rho\,(\partial_t I + V_1 \partial_1 I + V_2 \partial_2 I), \\ &c_{53} = 2\sin\phi, \quad c_{55} = -2Q_D \cos 2\,(\Psi_D - \Psi_\Sigma), \end{aligned} \tag{83}$$

where

$$Q_D = \frac{1}{2}\left[(D_{11} - D_{22})^2 + 4D_{12}^2\right]^{1/2}, \tag{84}$$

$$\tan 2\Psi_D = \frac{2D_{12}}{D_{11} - D_{22}}, \tag{85}$$

i.e., Q_D and Ψ_D are the maximum shear-rate and angle that the greater principal direction of the deformation-rate tensor makes with the x_1-axis, in the underlying unperturbed field, respectively. Let the entries of $B_1(\mathbf{Z})$, $B_2(\mathbf{Z})$ be denoted by B^1_{ij}, B^2_{ij}, respectively; then, since $\nu = 0$, and renumbering the rows and columns as necessary, we obtain from Equations (57), (58)

$$\begin{aligned}
&B^1_{14} = 1 - \sin\phi\cos 2\Psi_\Sigma, \quad B^1_{15} = 2Q_\Sigma \sin 2\Psi_\Sigma,\\
&B^1_{24} = -\sin\phi\sin 2\Psi_\sigma, \quad B^1_{25} = -2Q_\Sigma\cos 2\Psi_\Sigma,\\
&B^1_{41} = 1, \quad B^1_{42} = 0,\\
&B^1_{51} = -\sin 2\Psi_\Sigma, \quad B^1_{52} = -\sin\phi + \cos 2\Psi_\Sigma,\\
&B^2_{14} = -\sin\phi\sin 2\Psi_\Sigma, \quad B^2_{15} = -2Q_\Sigma\cos 2\Psi_\Sigma,\\
&B^2_{24} = 1 + \sin\phi\cos 2\Psi_\Sigma, \quad B^2_{25} = -2Q_\Sigma \sin 2\Psi_\Sigma,\\
&B^2_{41} = 0, \quad B^2_{42} = 1,\\
&B^2_{51} = \sin\phi, \quad B^2_{52} = \sin 2\Psi_\Sigma,
\end{aligned} \tag{86}$$

where

$$Q_\Sigma = \frac{1}{2}\left[(\Sigma_{11} - \Sigma_{22})^2 + 4\Sigma^2_{12}\right]^{1/2}, \tag{87}$$

$$\tan 2\Psi_\Sigma = \frac{\Sigma_{21} + \Sigma_{12}}{\Sigma_{11} - \Sigma_{22}}.$$

6. Method of frozen coefficients for the incompressible case

We now apply the method of frozen coefficients to show that the model is linearly well-posed for incompressible flows. Consider a perturbation $\mathbf{z}'$ of the original solution $\mathbf{Z}$, with initial time t_0 in the neighbourhood of the point $\mathbf{x}_0$ in which $\mathbf{z}'$ is a normal mode solution of the linearised Equations (81), *i.e.*,

$$\mathbf{z}' = \mathbf{z_0}\exp[\varsigma(t - t_0) + \mathrm{i}\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}_0)] \tag{88}$$

where

$$\mathbf{z^t_0} = \left(v^0_1, v^0_2, \omega^0, p^0_\sigma, \psi^0_\sigma\right) \tag{89}$$

denotes the initial amplitude of the perturbation, ς denotes the (possibly complex) frequency, $\mathrm{i} = \sqrt{-1}$, $\mathbf{k} = (k_1, k_2)$ the (real) vector wave-number, $\mathbf{x} = (x_1, x_2)$, t_0 denotes the initial time and $\mathbf{x}_0$ denotes a fixed spatial point in $\mathcal{R}$. Now,

$$\partial_t\mathbf{z}' = \varsigma\mathbf{z}', \qquad \partial_1\mathbf{z}' = \mathrm{i}k_1\mathbf{z}', \qquad \partial_2\mathbf{z}' = \mathrm{i}k_2\mathbf{z}', \tag{90}$$

and substituting in Equation (81) gives the homogeneous set of algebraic linear equations for $\mathbf{z_0}$,

$$[A\varsigma + E(\mathbf{k}, \mathbf{Z})]\mathbf{z_0} = 0, \tag{91}$$

where E denotes the matrix

$$E(\mathbf{k}, \mathbf{Z}) = \mathrm{i}k_1 B_1(\mathbf{Z}) + \mathrm{i}k_2 B_2(\mathbf{Z}) + C(\mathbf{Z}) \tag{92}$$

$$= \begin{bmatrix} e_{11} & c_{12} & 0 & e_{14} & e_{15} \\ c_{21} & e_{22} & 0 & e_{24} & e_{25} \\ c_{31} & c_{32} & e_{33} & 0 & 0 \\ e_{41} & e_{42} & 0 & 0 & 0 \\ e_{51} & e_{52} & c_{53} & 0 & c_{55} \end{bmatrix} \tag{93}$$

and where

$$e_{11} = \rho_0 e + c_{11}, \quad e_{22} = \rho_0 e + c_{22}, \quad e_{33} = \rho_0 I e + c_{33}, \tag{94}$$

$$e = \mathrm{i}k_1 V_1 + \mathrm{i}k_2 V_2, \tag{95}$$

$$\begin{aligned} e_{ij} &= \mathrm{i}k_1 B^1_{ij} + \mathrm{i}k_2 B^2_{ij} + c_{ij}, \quad i = 1, 2, j = 4, 5 \\ e_{ij} &= \mathrm{i}k_1 B^1_{ij} + \mathrm{i}k_2 B^2_{ij}, \quad i = 4, 5, j = 1, 2. \end{aligned} \tag{96}$$

Thus, each e_{ij} is linear in i$\mathbf{k}$, while c_{ij} is independent of $\mathbf{k}$. Recall that the symbol E was used for a related, but different, matrix in the section demonstrating hyperbolicity of the steady state equations. The similarities and differences between the two matrices are worth noting. The condition that Equation (91) gives rise to non-trivial solutions for $\mathbf{z_0}$, is

$$\det[A\varsigma + E(\mathbf{k}, \mathbf{Z})] = 0. \tag{97}$$

Now, the matrix A is singular and Equation (91) represents a generalised eigenvalue problem for ς,[1] and care must be taken to ensure that all possibilities for non-trivial solutions are found. Accordingly, we first reduce Equation (91) to a standard eigenvalue problem by using the fourth and fifth equations to eliminate the unknowns $p^0_\sigma, \psi^0_\sigma$. However, it should be pointed out that a direct expansion of the determinant in Equation (97) gives the same results concerning well-posedness as those given below, and, moreover, the calculation is shorter. Now, the fourth row of Equation (91) states

$$e_{41} v^0_1 + e_{42} v^0_2 = 0. \tag{98}$$

Multiplying the first equation by e_{41}, the second by e_{42} and adding gives

$$\begin{aligned} &(e_{11}e_{41} + e_{42}c_{21})\, v^0_1 + (e_{41}c_{12} + e_{42}e_{22})\, v^0_2 \\ &\quad + (e_{41}e_{14} + e_{42}e_{24})\, p^0_\sigma + (e_{41}e_{15} + e_{42}e_{25})\, \psi^0_\sigma = 0. \end{aligned} \tag{99}$$

The fifth row of Equation (91) states

$$e_{51} v^0_1 + e_{52} v^0_2 + c_{53}\omega^0 + c_{55}\psi^0_\sigma = 0.$$

Multiplying the first equation by e_{51}, the second by e_{52} and adding gives

$$\begin{aligned} &(e_{11}e_{51} + c_{21}e_{52})\, v^0_1 + (c_{12}e_{51} + e_{22}e_{52})\, v^0_2 - \rho_0\varsigma c_{53}\omega^0 \\ &\quad + (e_{15}e_{51} + e_{25}e_{52})\, p^0_\sigma + (e_{15}e_{51} + e_{25}e_{52} - c_{55}\rho_0\varsigma)\, \psi^0_\sigma = 0. \end{aligned} \tag{100}$$

[1]One of the authors (DH) is indebted for remarks by Prof. D.G. Schaeffer on this point.

Equations (99) and (100) may be solved for p_σ^0, ψ_σ^0,

$$p_\sigma^0 = \frac{(f_{41} + g_{41}\rho_0\varsigma)\, v_1^0 + (f_{42} + g_{42}\rho_0\varsigma)\, v_2^0 + g_{43}\rho_0\varsigma\omega^0}{f + g\rho_0\varsigma}, \tag{101}$$

$$\psi_\sigma^0 = \frac{f_{51} v_1^0 + f_{52} v_2^0 + g_{53}\rho_0\varsigma\omega^0}{f + g\rho_0\varsigma}, \tag{102}$$

where

$$f = (e_{14}e_{25} - e_{15}e_{24})\,(e_{42}e_{51} - e_{41}e_{52})\,, \quad g = c_{55}\,(e_{41}e_{14} + e_{42}e_{24}) \tag{103}$$

and these are the key quantities in determining the linear well-posedness of the system. Also

$$\begin{aligned} f_{41} &= (c_{21}e_{15} - e_{11}e_{25})\,(e_{42}e_{51} - e_{41}e_{52})\,,\\ f_{42} &= (e_{15}e_{22} - e_{25}c_{12})\,(e_{42}e_{51} - e_{41}e_{52})\,,\\ f_{51} &= (e_{15}c_{21} - e_{25}e_{11})\,(e_{41}e_{52} - e_{42}e_{51})\,,\\ f_{52} &= (e_{14}e_{22} - e_{24}c_{12})\,(e_{41}e_{52} - e_{42}e_{51})\,, \end{aligned} \tag{104}$$

$$\begin{aligned} g_{41} &= -c_{55}\,(e_{11}e_{41} + e_{42}c_{21})\,, \quad & g_{42} &= -c_{55}\,(e_{41}c_{12} + e_{42}e_{22})\,,\\ g_{43} &= c_{53}\,(e_{41}e_{14} + e_{42}e_{24})\,, & g_{43} &= -c_{53}\,(e_{41}e_{14} + e_{42}e_{24})\,. \end{aligned} \tag{105}$$

Elimination of $p_\sigma^0, \psi_\sigma^0$ from the equations and defining $\mathbf{z}_r^0 = \left(v_1^0, v_2^0, \omega^0\right)$ reduces the system of equations to

$$\begin{bmatrix} g\rho_0^2\varsigma^2 + h_{11}\rho_0\varsigma + i_{11} & h_{12}\rho_0\varsigma + i_{12} & h_{13}\rho_0\varsigma \\ h_{21}\rho_0\varsigma + i_{21} & g\rho_0^2\varsigma^2 + h_{22}\rho_0\varsigma + i_{22} & h_{23}\rho_0\varsigma \\ c_{31} & c_{32} & \rho_0 I\varsigma + e_{33} \end{bmatrix} \begin{bmatrix} v_1^0 \\ v_2^0 \\ \omega^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{106}$$

where

$$\begin{aligned} h_{11} &= f + e_{11}g + e_{15}g_{41}, & h_{12} &= c_{12}g + e_{15}g_{42},\\ h_{13} &= e_{15}g_{43} + e_{16}g_{53}, & h_{21} &= c_{21}g + e_{25}g_{41},\\ h_{22} &= f + e_{22}g + e_{25}g_{42}, & h_{23} &= e_{25}g_{43} + e_{26}g_{53},\\ i_{11} &= e_{11}f + e_{15}f_{41} + e_{16}f_{51}, & i_{12} &= c_{12}f + e_{15}f_{42} + e_{16}f_{52},\\ i_{21} &= c_{21}f + e_{26}f_{51} + e_{25}f_{41}, & i_{22} &= e_{22}f + e_{25}f_{42} + e_{26}f_{52}. \end{aligned} \tag{107}$$

A direct calculation shows that

$$i_{11} = i_{12} = i_{21} = i_{22} = 0 \tag{108}$$

and so the equations reduce to

$$F(\varsigma, \mathbf{k}, \mathbf{Z})\mathbf{z}_r^0 = 0, \tag{109}$$

where

$$F(\varsigma, \mathbf{k}, \mathbf{Z}) = \begin{bmatrix} g\rho_0^2\varsigma^2 + h_{11}\rho_0\varsigma & h_{12}\rho_0\varsigma & h_{13}\rho_0\varsigma \\ h_{21}\rho_0\varsigma & g\rho_0^2\varsigma^2 + h_{22}\rho_0\varsigma & h_{23}\rho_0\varsigma \\ c_{31} & c_{32} & \rho_0 I\varsigma + e_{33} \end{bmatrix}. \tag{110}$$

This is now a standard eigenvalue problem and Equation (109) has non-trivial solutions for $\mathbf{z}_r^0$ provided

$$\det F(\varsigma, \mathbf{k}, \mathbf{Z}) = 0 \tag{111}$$

and equation (111), the *dispersion relation* for the model, determines ς in terms of $\mathbf{k}$ and $\mathbf{Z}$,

$$\varsigma = \varsigma(\mathbf{k}, \mathbf{Z}). \tag{112}$$

Bearing in mind Equation (88), the *growth rate* of the model, for given $\mathbf{k}$, $\mathbf{Z}$ is defined to be the quantity $\mathfrak{Re}(\varsigma)$ where $\mathfrak{Re}$ denotes real part. It is convenient to define the *wave number tensor* $\mathbf{K}$ by

$$\mathbf{K} = \mathbf{k} \otimes \mathbf{k} = \begin{bmatrix} k_1^2 & k_1 k_2 \\ k_1 k_2 & k_2^2 \end{bmatrix} \tag{113}$$

with invariant k, where

$$k = \sqrt{k_1^2 + k_2^2}, \tag{114}$$

and the *wave tensor angle* ψ_K by

$$\tan \psi_K = k_2 / k_1. \tag{115}$$

Then

$$k_1 = k \cos \psi_K, \quad k_2 = k \sin \psi_K. \tag{116}$$

For $k \neq 0$ define

$$\bar{k}_i = k_i / k. \tag{117}$$

The relations

$$\sin 2\psi_K = \frac{2k_1 k_2}{k_1^2 + k_2^2} = 2\bar{k}_1 \bar{k}_2, \tag{118}$$

$$\cos 2\psi_K = \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2} = \bar{k}_1^2 - \bar{k}_2^2. \tag{119}$$

will be particularly useful. Then

$$\det F(\varsigma, k, \psi_K, \mathbf{Z}) = 0 \tag{120}$$

and expanding the dispersion relation (120) using Equation (110) gives the following quintic equation in ς

$$A_3 (\rho_0 \varsigma)^5 + A_2 (\rho_0 \varsigma)^4 + A_1 (\rho_0 \varsigma)^3 + A_0 (\rho_0 \varsigma)^2 = 0, \tag{121}$$

in which the coefficients $A_i = A_i(k, \psi_K, \mathbf{Z})$ are

$$\begin{aligned} A_3 &= I g^2, \\ A_2 &= g\left[e_{33} g + I(h_{11} + h_{22})\right], \\ A_1 &= I(h_{11}h_{22} - h_{12}h_{21}) + e_{33} g(h_{11} + h_{22}) - g(c_{31}h_{13} + c_{32}h_{23}), \\ A_0 &= e_{33}(h_{11}h_{22} - h_{12}h_{21}) + c_{32}(h_{13}h_{21} - h_{11}h_{23}) + c_{31}(h_{12}h_{23} - h_{13}h_{22}), \end{aligned} \tag{122}$$

where, from Equations (107)

$$h_{11}+h_{22}=2f+c_{55}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right], \tag{123}$$

$$h_{11}h_{22}-h_{12}h_{21}=f^2+c_{55}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]f, \tag{124}$$

$$c_{31}h_{13}+c_{32}h_{23}=c_{53}\left(e_{14}e_{25}-e_{15}e_{24}\right)\left(c_{31}e_{42}-c_{32}e_{41}\right), \tag{125}$$

$$\begin{aligned} &c_{32}\left(h_{13}h_{21}-h_{11}h_{23}\right)+c_{31}\left(h_{12}h_{23}-h_{13}h_{22}\right)\\ &\quad =c_{53}\left(c_{32}b_{41}-c_{31}b_{42}\right)\left(e_{14}e_{25}-e_{15}e_{24}\right)f. \end{aligned} \tag{126}$$

Thus,

$$A_2=2Ifg+e_{33}g^2+Ic_{55}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]g, \tag{127}$$

$$\begin{aligned} A_1=If^2&+\{Ic_{55}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]+2e_{33}g\}f\\ &+\{c_{55}e_{33}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]\\ &-c_{53}\left(c_{31}e_{42}-c_{32}e_{41}\right)\left(e_{14}e_{25}-e_{15}e_{24}\right)\}g, \end{aligned} \tag{128}$$

$$\begin{aligned} A_0=\{&e_{33}\{f+c_{55}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]\}\\ &+c_{53}\left(c_{32}e_{41}-c_{31}e_{42}\right)\left(e_{14}e_{25}-e_{15}e_{24}\right)\}f. \end{aligned} \tag{129}$$

Now, $\varsigma=\varsigma\left(k,\psi_K,\mathbf{Z}\right)$ and we define

$$\varsigma_\infty=\varsigma_\infty\left(\psi_K,\mathbf{Z}\right)=\lim_{k\to\infty}\varsigma\left(k,\psi_K,\mathbf{Z}\right). \tag{130}$$

The quantity $\mathfrak{Re}\left(\varsigma_\infty\right)$ is called the *asymptotic growth rate* of the model. The model is called *linearly well-posed* if $\mathfrak{Re}\left(\varsigma_\infty\right)$ is finite for all values of ψ_K. On the other hand, it is called *linearly ill-posed* if $\mathfrak{Re}\left(\varsigma_\infty\right)\to+\infty$ as $k\to\infty$ for some value of ψ_K. Thus, a well-posed model may admit unstable solutions but the growth rate ς must be bounded in its dependence on k. But then, in a linearly ill-posed model, ς is unbounded as a function of k, so, the shorter the wavelength of the perturbation, the larger its growth rate, pointing to a particularly strong kind of instability. Of course, these growing perturbations are solutions of only the linearised equations of the model, which become invalid as the perturbations grow. In the full model, the strength of the growth of perturbations may be mitigated by the nonlinearities. However, for the quasi-linear models considered here, a (nonlinearly) ill-posed model cannot give rise to a well-posed linearisation, and a linearly ill-posed model cannot be the linearisation of a (nonlinearly) well-posed model; see Strang [24]. It is in this sense that the linearised analysis presented here gives a valid deduction for the full quasi-linear model. Our goal is to evaluate $\mathfrak{Re}\left(\varsigma_\infty\right)$ for all values of ψ_K and this we will do by finding the asymptotic growth rate of the roots of Equation (121).

7. Linear well-posedness for normal directions

From Equation (121) either $\varsigma=0$ is a repeated root, corresponding to a neutral affect on growth or decay, or

$$A_3\left(\rho_0\varsigma\right)^3+A_2\left(\rho_0\varsigma\right)^2+A_1\rho_0\varsigma+A_0=0. \tag{131}$$

By inspection, from Equations $(122)_1$, (127–129), we have

$$\begin{aligned} &A_3\left(\mathbf{k},\mathbf{Z}\right)=O\left(k^4\right), \quad A_2\left(\mathbf{k},\mathbf{Z}\right)=O\left(k^6\right),\\ &A_1\left(\mathbf{k},\mathbf{Z}\right)=O\left(k^8\right), \quad A_0\left(\mathbf{k},\mathbf{Z}\right)=O\left(k^9\right). \end{aligned} \tag{132}$$

Let the roots of the dispersion relation, Equation (131), be denoted by α, β, γ; then

$$\alpha+\beta+\gamma=O\left(k^{2}\right), \quad \alpha\beta+\alpha\gamma+\beta\gamma=O\left(k^{4}\right), \quad \alpha\beta\gamma=O\left(k^{5}\right). \tag{133}$$

Hence, there are two roots $O\left(k^{2}\right)$ and one root $O\left(k\right)$. We now investigate the asymptotic behaviour of the roots α, β, γ.

7.1. The roots $O\left(k^{2}\right)$

To investigate the $O\left(k^{2}\right)$ roots, we define

$$\bar{\varsigma}=\bar{\varsigma}\left(k, \psi_K, \mathbf{Z}\right)=\varsigma/k^{2}, \quad \bar{e}_{ij}=e_{ij}/k, \quad \bar{c}_{ij}=c_{ij}/k. \tag{134}$$

Also let

$$\bar{f}=\left(\bar{e}_{14}\bar{e}_{25}-\bar{e}_{15}\bar{e}_{24}\right)\left(\bar{e}_{42}\bar{e}_{51}-\bar{e}_{41}\bar{e}_{52}\right), \qquad \bar{g}=c_{55}\left(\bar{e}_{41}\bar{e}_{14}+\bar{e}_{42}\bar{e}_{24}\right). \tag{135}$$

We now consider the behaviour of $\mathfrak{Re}\left(\bar{\varsigma}\right)$ in the limit $k\to\infty$ with ψ_K held constant. Let $\bar{\varsigma}_{\infty}=\lim_{k\to\infty}\bar{\varsigma}$ then $\mathfrak{Re}\left(\bar{\varsigma}_{\infty}\right)$ determines the asymptotic growth rate of the model since

$$\bar{\varsigma}_{\infty}=\lim_{k\to\infty}\varsigma_{\infty}/k^{2}. \tag{136}$$

Clearly, for a $O\left(k^{2}\right)$ root, the model is linearly well-posed if, and only if, $\bar{\varsigma}_{\infty}\leq 0$. We re-scale the dispersion relation to render the $O\left(k^{2}\right)$ root finite in the limit $k\to\infty$ and use the scaled dispersion relation to obtain the value of $\mathfrak{Re}\left(\varsigma_{\infty}\right)$. Dividing Equation (131) by k^{10}, we may write the dispersion relation as follows:

$$\bar{A}_3\left(\rho_0\bar{\varsigma}\right)^{3}+\bar{A}_2\left(\rho_0\bar{\varsigma}\right)^{2}+\bar{A}_1\rho_0\bar{\varsigma}+\bar{A}_0/k=0, \tag{137}$$

where $\bar{A}_3=\bar{A}_3\left(\psi_K, \mathbf{Z}\right)$, all other $\bar{A}_i=\bar{A}_i\left(k, \psi_K, \mathbf{Z}\right)$ and

$$\begin{aligned}
\bar{A}_3&=I\bar{g}^{2},\\
\bar{A}_2&=2I\bar{f}\bar{g}+\tfrac{1}{k}\bar{e}_{33}\bar{g}^{2}+I\bar{c}_{55}\left[\bar{e}_{42}\left(\bar{e}_{11}\bar{e}_{24}-\bar{e}_{14}\bar{c}_{21}\right)+\bar{e}_{41}\left(\bar{e}_{22}\bar{e}_{14}-\bar{e}_{24}\bar{c}_{12}\right)\right]\bar{g},\\
\bar{A}_1&=I\bar{f}^{2}+\left\{I\bar{c}_{55}\left[\bar{e}_{52}\left(\bar{e}_{11}\bar{e}_{24}-\bar{e}_{14}\bar{c}_{21}\right)+\bar{e}_{41}\left(\bar{e}_{22}\bar{e}_{14}-\bar{e}_{24}\bar{c}_{12}\right)\right]+\tfrac{2}{k}\bar{e}_{33}\bar{g}\right\}\bar{f}\\
&\quad+\tfrac{1}{k}\left\{\bar{c}_{55}\bar{e}_{33}\left[\bar{e}_{42}\left(\bar{e}_{11}\bar{e}_{24}-\bar{e}_{14}\bar{c}_{21}\right)+\bar{e}_{41}\left(\bar{e}_{22}\bar{e}_{14}-\bar{e}_{25}\bar{c}_{12}\right)\right]\right.\\
&\quad\left.-\bar{c}_{53}\left(\bar{c}_{31}\bar{e}_{42}-\bar{c}_{32}\bar{e}_{41}\right)\left(\bar{e}_{14}\bar{e}_{25}-\bar{e}_{15}\bar{e}_{24}\right)\right\}\bar{g},\\
\bar{A}_0&=\left\{\bar{e}_{33}\left\{\bar{f}+\bar{c}_{55}\left[\bar{e}_{42}\left(\bar{e}_{11}\bar{e}_{24}-\bar{e}_{14}\bar{c}_{21}\right)+\bar{e}_{41}\left(\bar{e}_{22}\bar{e}_{14}-\bar{e}_{24}\bar{c}_{12}\right)\right]\right\}\right.\\
&\quad\left.+\bar{c}_{53}\left(\bar{c}_{32}\bar{e}_{41}-\bar{c}_{31}\bar{e}_{42}\right)\left(\bar{e}_{14}\bar{e}_{25}-\bar{e}_{15}\bar{e}_{24}\right)\right\}\bar{f}.
\end{aligned} \tag{138}$$

Let $k\to\infty$ with ψ_K held constant and define

$$\bar{e}_{ij}^{\infty}=\lim_{k\to\infty}e_{ij}/k, \tag{139}$$

$$\bar{f}_{\infty}=\left(\bar{e}_{14}^{\infty}\bar{e}_{25}^{\infty}-\bar{e}_{15}^{\infty}\bar{e}_{24}\right)\left(\bar{e}_{42}^{\infty}\bar{e}_{51}^{\infty}-\bar{e}_{41}^{\infty}\bar{e}_{52}^{\infty}\right), \quad \bar{g}_{\infty}=c_{55}\left(\bar{e}_{41}^{\infty}\bar{e}_{14}^{\infty}+\bar{e}_{42}^{\infty}\bar{e}_{24}^{\infty}\right), \tag{140}$$

noting that

$$\lim_{k\to\infty}\bar{c}_{ij}=0. \tag{141}$$

Then the dispersion relation reduces to the following cubic equation, the *first reduced asymptotic dispersion relation* for $\bar{\varsigma}_{\infty}$,

$$\bar{A}_3^{\infty}\left(\rho_0\bar{\varsigma}_{\infty}\right)^{3}+\bar{A}_2^{\infty}\left(\rho_0\bar{\varsigma}_{\infty}\right)^{2}+\bar{A}_1^{\infty}\rho_0\bar{\varsigma}_{\infty}=0, \tag{142}$$

where

$$\bar{A}_i^{\infty} = \bar{A}_i^{\infty}(\psi_K, \mathbf{Z}) = \lim_{k\to\infty} \bar{A}_i \tag{143}$$

and

$$\bar{A}_3^{\infty} = I\bar{g}_{\infty}^2, \quad \bar{A}_2^{\infty} = 2I\bar{f}_{\infty}\bar{g}_{\infty}, \quad \bar{A}_1^{\infty} = I\bar{f}_{\infty}^2, \quad \bar{A}_0^{\infty} = \bar{e}_{33}\bar{f}_{\infty}^2. \tag{144}$$

From Equation (142), either $\bar{\varsigma}_{\infty} = 0$ (which corresponds to the $O(k)$ bounded root of Equation (131)) or

$$\left(\rho_0\bar{g}_{\infty}\bar{\varsigma}_{\infty} + \bar{f}_{\infty}\right)^2 = 0, \tag{145}$$

i.e.,

$$\bar{\varsigma}_{\infty} = -\frac{\left(\bar{e}_{14}^{\infty}\bar{e}_{25}^{\infty} - \bar{e}_{15}^{\infty}\bar{e}_{24}\right)\left(\bar{e}_{42}^{\infty}\bar{e}_{51}^{\infty} - \bar{e}_{41}^{\infty}\bar{e}_{52}^{\infty}\right)}{\rho_0 c_{55}\left(\bar{e}_{41}^{\infty}\bar{e}_{14}^{\infty} + \bar{e}_{42}^{\infty}\bar{e}_{24}^{\infty}\right)}. \tag{146}$$

Now,

$$\bar{e}_{ij}^{\infty} = \mathrm{i}\bar{k}_1 B_{ij}^1 + \mathrm{i}\bar{k}_2 B_{ij}^2, \tag{147}$$

and hence, on using Equations (147), (86), (118), (119) and (19), we have

$$\begin{aligned}
\bar{e}_{14}^{\infty}\bar{e}_{25}^{\infty} - \bar{e}_{15}^{\infty}\bar{e}_{24} &= 2Q_{\Sigma}\left[\cos 2\left(\Psi_{\Sigma} - \psi_K\right) - \sin\phi\right], \\
&= 4Q_{\Sigma}\cos\left(\Psi_{\Sigma} + \varepsilon - \psi_K\right)\cos\left(\Psi_{\Sigma} - \varepsilon - \psi_K\right), \\
\bar{e}_{42}^{\infty}\bar{e}_{51}^{\infty} - \bar{e}_{41}^{\infty}\bar{e}_{52}^{\infty} &= \left[\cos 2\left(\Psi_{\Sigma} - \psi_K\right) - \sin\phi\right], \\
&= 2\cos\left(\Psi_{\Sigma} + \varepsilon - \psi_K\right)\cos\left(\Psi_{\Sigma} - \varepsilon - \psi_K\right), \\
\bar{e}_{41}^{\infty}\bar{e}_{14}^{\infty} + \bar{e}_{42}^{\infty}\bar{e}_{24}^{\infty} &= -1 + \sin\phi\cos 2\left(\Psi_{\Sigma} - \psi_K\right).
\end{aligned} \tag{148}$$

Thus, using Equation $(83)_{\mathrm{ix}}$ yields

$$\bar{\varsigma}_{\infty} = -\frac{Q_{\Sigma}\left[\cos 2\left(\Psi_{\Sigma} - \psi_K\right) - \sin\phi\right]^2}{\rho_0 Q_D \cos 2\left(\Psi_D - \Psi_{\Sigma}\right)\left[1 - \sin\phi\cos 2\left(\Psi_{\Sigma} - \psi_K\right)\right]} \tag{149}$$

$$= -\frac{4Q_{\Sigma}\sin^2\left(\Psi_{\Sigma} - \psi_K + \frac{1}{4}\pi - \frac{1}{2}\phi\right)\sin^2\left(\Psi_{\Sigma} - \psi_K - \frac{1}{4}\pi + \frac{1}{2}\phi\right)^2}{\rho_0 Q_D \cos 2\left(\Psi_D - \Psi_{\Sigma}\right)\left[1 - \sin\phi\cos 2\left(\Psi_{\Sigma} - \psi_K\right)\right]}. \tag{150}$$

Now, for all values of ψ_K

$$1 - \sin\phi\cos 2\left(\Psi_{\Sigma} - \psi_K\right) > 0, \tag{151}$$

provided $0 < \phi < \frac{1}{2}\pi$ and so $\bar{\varsigma}_{\infty} \leq 0$ provided the following bound on non-coaxiality in the underlying prescribed solution is satisfied:

$$-\pi/4 < \Psi_D - \Psi_{\Sigma} < \pi/4. \tag{152}$$

We note from its derivation, that the asymptotic equation (142) fails to be valid when

$$\left(\bar{e}_{14}\bar{e}_{25} - \bar{e}_{15}\bar{e}_{24}\right)\left(\bar{e}_{42}\bar{e}_{51} - \bar{e}_{41}\bar{e}_{52}\right) = 0, \tag{153}$$

since terms in Equation (131) which have been omitted from (142) on the condition that they may be neglected in comparison with the term on the left-hand side of Equation (153) can no longer be so when this term is identically zero. Equation (153) holds when ψ_K takes one of the values ψ_K^1, ψ_K^2, where

$$\begin{aligned}\psi_K^1 &= \Psi_\Sigma - \pi/4 - \phi/2 \pm \pi/2 = \Psi_\Sigma - \varepsilon \pm \pi/2,\\ \psi_K^2 &= \Psi_\Sigma + \pi/4 + \phi/2 \pm \pi/2 = \Psi_\Sigma + \varepsilon \pm \pi/2,\end{aligned} \tag{154}$$

i.e., the directions normal to the coincident stress and velocity characteristic directions of the underlying prescribed solution. A *degenerate direction in the* (k_1,k_2)*-plane* corresponds to one of the values ψ_K^1, ψ_K^2 of ψ_K given by Equation (154). All other directions in the (k_1,k_2)-plane are called *normal directions.* Thus, we have shown that the two roots $O(k^2)$ give a well-posed contribution in the normal directions.

7.2. The root $O(k)$

We now consider the remaining root of Equation (131) subject to Equations (132). To investigate this root we again re-scale the dispersion relation, this time to render the $O(k)$ root finite in the limit $k \to \infty$ and use the scaled dispersion relation to obtain the value of ς_∞. We now divide Equation (131) by k^9 to obtain the dispersion relation

$$\frac{1}{k^2}\bar{A}_3\,(\rho_0\bar{\varsigma})^3 + \frac{1}{k}\bar{A}_2\,(\rho_0\bar{\varsigma})^2 + \bar{A}_1\rho_0\bar{\varsigma} + \bar{A}_0 = 0, \tag{155}$$

where the $\bar{A}_i$ are given by Equation (138). Let $k \to \infty$ with ψ_K held constant then the dispersion relation reduces to the following linear equation, the *second reduced asymptotic dispersion relation* for $\bar{\varsigma}_\infty$,

$$\bar{A}_1^\infty \rho_0 \bar{\varsigma}_\infty + \bar{A}_0^\infty = 0, \tag{156}$$

where the $\bar{A}_i^\infty$ are given by Equation (144). Thus

$$\bar{\varsigma}_\infty = -\bar{e}_\infty, \tag{157}$$

where

$$\bar{e}_\infty = \mathrm{i}\,(V_1 \cos\psi_k + V_2 \sin\psi_k)\,, \tag{158}$$

which is purely imaginary, $\mathfrak{Re}\,(\bar{\varsigma}_\infty) = 0$, and hence this root cannot contribute to ill-posedness. Again we see that the reduced dispersion relation ceases to be valid in a direction perpendicular to the corresponding characteristic, in this case the streamline characteristic, *i.e.*, the second reduced asymptotic dispersion relation ceases to be valid in the direction

$$\psi_K^3 = \tan^{-1}(-V_1/V_2)\,. \tag{159}$$

7.3. Degenerate directions

We now turn to the values of ψ_K that correspond to the degenerate directions, ψ_K^1, ψ_K^2, and also to the direction ψ_K^3, *i.e.*, the directions normal to each of the three distinct characteristic directions. As stated above, the asymptotic equations (142) break down for $\psi_k = \psi_K^1, \psi_K^2$. In this case

$$k_1 = \pm k \sin(\Psi_\sigma \pm \varepsilon)\,, \quad k_2 = \pm k \cos(\Psi_\sigma \pm \varepsilon)\,, \tag{160}$$

and we now derive asymptotic dispersion equations applicable to the degenerate directions. It seems intuitively clear that in the degenerate directions the terms which give rise to the roots

$O(k^2)$ are absent, thus leaving only the root $O(k)$ to determine the growth rate. We verify here that this is indeed the case. Now, let ψ_K take one of the values given by Equations (154); then

$$\begin{aligned} &\bar{e}_{14}^{\infty}\bar{e}_{25}^{\infty}-\bar{e}_{15}^{\infty}\bar{e}_{24}=\bar{e}_{42}^{\infty}\bar{e}_{51}^{\infty}-\bar{e}_{41}^{\infty}\bar{e}_{52}^{\infty}=0,\\ &\bar{e}_{41}^{\infty}\bar{e}_{14}^{\infty}+\bar{e}_{42}^{\infty}\bar{e}_{24}^{\infty}=-\cos^2\phi. \end{aligned} \tag{161}$$

since $\cos 2\left(\Psi_\sigma-\psi_K\right)=\sin\phi$. Also, Equations $(122)_1$, (127–129) reduce to

$$\begin{aligned} &A_3=Ig^2\\ &A_2=e_{33}g^2+Ic_{55}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]g,\\ &A_1=\{c_{55}e_{33}\left[e_{42}\left(e_{11}e_{24}-e_{14}c_{21}\right)+e_{41}\left(e_{22}e_{14}-e_{24}c_{12}\right)\right]\\ &\qquad -c_{53}\left(c_{31}e_{42}-c_{32}e_{41}\right)\left(e_{14}e_{25}-e_{15}e_{24}\right)\}\,g,\\ &A_0=0. \end{aligned} \tag{162}$$

Hence

$$\left[A_3\left(\rho_0\varsigma\right)^2+A_2\left(\rho_0\varsigma\right)+A_1\right]\left(\rho_0\varsigma\right)^3=0, \tag{163}$$

i.e., either $\varsigma=0$, a triple root, or

$$A_3\left(\rho_0\varsigma\right)^2+A_2\left(\rho_0\varsigma\right)+A_1=0, \tag{164}$$

where

$$A_3=O\left(k^4\right),\quad A_2=O\left(k^5\right),\quad A_1=O\left(k^6\right). \tag{165}$$

Denoting the roots of the quadratic equation (164) by α,β,

$$\alpha+\beta=O\left(k\right),\quad \alpha\beta=O\left(k^2\right) \tag{166}$$

i.e., there are two roots $O\left(k\right)$. Defining

$$\bar{\varsigma}=\varsigma/k \tag{167}$$

and dividing the equation by k^6 gives

$$\bar{A}_3\left(\rho_0\bar{\varsigma}\right)^2+\bar{A}_2\left(\rho_0\bar{\varsigma}\right)+\bar{A}_1=0, \tag{168}$$

where

$$\begin{aligned} &\bar{A}_2=\bar{e}_{33}\bar{g}^2+Ic_{55}\left[\bar{e}_{42}\left(\bar{e}_{11}\bar{e}_{24}-\bar{e}_{14}\bar{c}_{21}\right)+\bar{e}_{41}\left(\bar{e}_{22}\bar{e}_{14}-\bar{e}_{24}\bar{c}_{12}\right)\right]\bar{g},\\ &\bar{A}_1=\{c_{55}\bar{e}_{33}\left[\bar{e}_{42}\left(\bar{e}_{11}\bar{e}_{24}-\bar{e}_{14}\bar{c}_{21}\right)+\bar{e}_{41}\left(\bar{e}_{22}\bar{e}_{14}-\bar{e}_{24}\bar{c}_{12}\right)\right]\\ &\qquad -c_{53}\left(\bar{c}_{31}\bar{e}_{42}-\bar{c}_{32}\bar{e}_{41}\right)\left(\bar{e}_{14}\bar{e}_{25}-\bar{e}_{15}\bar{e}_{24}\right)\}\,\bar{g}. \end{aligned} \tag{169}$$

Letting $k\to\infty$ gives

$$\bar{A}_3^{\infty}=I\bar{g}_{\infty}^2,\quad \bar{A}_2^{\infty}=2\rho_0I\bar{e}_{\infty}\bar{g}_{\infty}^2,\quad \bar{A}_1^{\infty}=I\rho_0^2\bar{e}_{\infty}^2\bar{g}_{\infty}^2$$

and the *third reduced asymptotic dispersion relation* is

$$\bar{\varsigma}_{\infty}^2+2\bar{e}_{\infty}\bar{\varsigma}_{\infty}+\bar{e}_{\infty}^2=0 \tag{170}$$

with repeated root

$$\bar{\varsigma}=-\bar{e}_{\infty}=-\mathrm{i}\left(V_1\cos\psi_K^j+V_2\sin\psi_K^j\right),$$

where j may take the values 1, 2 and again this purely imaginary root cannot cause ill-posedness.

Finally, we turn to the direction, $\psi_K^3 = \tan^{-1}(-V_1/V_2)$. If ψ_K^3 does not coincide with one of ψ_K^1, ψ_K^2 then ψ_K^3 is a normal direction and need not be considered further. If ψ_K^3 does coincide with one of ψ_K^1, ψ_K^2 then both $f = e = 0$, and so Equations (122)$_\mathrm{i}$, (127–129) reduce further to

$$\begin{aligned} &A_3 = Ig^2, \quad A_2 = c_{33}g^2 + Ic_{55}\left[e_{42}(c_{11}e_{24} - e_{14}c_{21}) + e_{41}(c_{22}e_{14} - e_{24}c_{12})\right]g, \\ &A_1 = c_{55}c_{33}\left[e_{42}(c_{11}e_{24} - e_{14}c_{21}) + e_{41}(c_{22}e_{14} - e_{24}c_{12})\right]g, \quad A_0 = 0, \end{aligned} \tag{171}$$

but then

$$A_3 = A_2 = A_1 = O(k^4) \tag{172}$$

and all the roots are bounded and hence cannot contribute to ill-posedness.

8. Conclusions and discussion

We have presented a rigid/perfectly plastic model for the flow of granular materials which is closely related to the double-shearing model, the double-sliding free-rotating model and the associated flow rule. The two essential results of the paper are that (a) the model has been shown to be hyperbolic for steady-state flows in two space dimensions, irrespective of whether the flow is quasi-static or dynamic, and (b) incompressible flows are well-posed.

We make some further remarks concerning the significance of the model and its properties. One way to regard the model is that the double-sliding free-rotating model is indeterminate, containing one more unknown (ω) than equations. The double-shearing model closes this set of equations by taking ω to be $\dot{\psi}_\sigma$, the material derivative of the angle, that the algebraically greater principal stress direction makes with the x_1-axis. The present model is an alternative method of closure, in which the model is augmented by a further unknown (r_σ) and further equations (the equation of rotational motion and the rotational yield condition).

For the double-shearing model, many of the analytic solutions that have been found are such that the quantity $\dot{\psi}_\sigma$, is zero. All of these solutions are also solutions of the present model in which the intrinsic spin ω is zero. It may be anticipated that these solutions will be of use in applications of the present model, and work is currently in progress on this. The model has been constructed purposefully to demonstrate the existence of a model which contains sufficient mechanical and kinematic properties to describe the major bulk properties of granular materials, which also contains a domain of well-posedness and which retains the property of remaining hyperbolic in the inertial regime. This is in contrast to both the double-shearing model and the plastic-potential model when a non-associated flow rule is used.

Hyperbolicity in both the quasi-static and inertial regimes is a desirable property. In a real granular material, the inertial terms can never be identically zero, even though they may be very small. A model which changes type from hyperbolic to elliptic in the presence of inertia has the following difficulty. Solutions for a hyperbolic model need not be smooth, discontinuities in the field variables or their spatial derivatives are very common, indeed one of the standard methods for obtaining solutions to the quasi-static stress field involves patching together solutions which are continuous in the stresses, but in which the spatial derivatives tangential to one family of characteristics, are discontinuous. Solutions to elliptic models exhibit much more smoothness, typically, they must be analytic functions. So for a model which changes type from hyperbolic to elliptic, it appears that solutions in the inertial regime must be of a completely different character to solutions in the quasi-static regime, even in the case where the inertial terms are arbitrarily small. This difference also holds true for boundary conditions. Solutions of hyperbolic models often exhibit discontinuities in tangential components of velocity and this describes well the slipping of a granular material over a bounding surface, or an internal slip surface. Elliptic models, on the other hand, cannot readily incorporate such discontinuities due to the required smoothness of the solution. Hence an elliptic model must resort to velocity boundary conditions such as the no-slip condition. However, it may be argued that the no-slip condition can never be assured to apply to granular materials. If a bounding surface be perfectly rough, so that no relative velocity is allowed (this does not, of course, preclude rolling) between the material comprising the boundary and the granular material itself, then the layer of grains in direct contact with the boundary will surely stick to it, but the next layer of grains further out *may indeed* slip on the inner layer. In this case it is often said that the material slips on itself. Since, in a continuum model, the grain size may be taken as zero, this slip line cannot be distinguished from the boundary itself and a tangential velocity discontinuity at the boundary is required.

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